3.6 Leading behavior and L’Hopital’s Rule

MATHEMATICAL TECHNIQUES

♠ Find the leading behavior of the following functions at 0 and ∞.

• EXERCISE 3.6.1
  \( f(x) = 1 + x. \)

• EXERCISE 3.6.2
  \( g(y) = y + y^3. \)

• EXERCISE 3.6.3
  \( h(z) = z + e^z. \)

• EXERCISE 3.6.4
  \( F(x) = 1 + 2x + 3e^x. \)

• EXERCISE 3.6.5
  \( m(a) = 100a + 30a^2 + \frac{1}{a}. \)

• EXERCISE 3.6.6
  \( G(c) = e^{-4c} + \frac{5}{c^2} + \frac{3}{c^6} + 10e^{-3c}. \)

♠ For each pair of functions, use the basic functions (when possible) to say which approaches its limit more quickly, and then check with L’Hopital’s rule.

• EXERCISE 3.6.7
  \( x^2 \) and \( e^{2x} \) as \( x \to \infty. \)

• EXERCISE 3.6.8
  \( x^2 \) and \( 1000x \) as \( x \to \infty. \)

• EXERCISE 3.6.9
  \( 0.1x^{0.5} \) and \( 30 \ln(x) \) as \( x \to \infty. \)

• EXERCISE 3.6.10
  \( x \) and \( \ln(x)^2 \) as \( x \to \infty. \)

• EXERCISE 3.6.11
  \( e^{-2x} \) and \( x^{-2} \) as \( x \to \infty. \)

• EXERCISE 3.6.12
  \( 1/\ln(x) \) and \( 30x^{-0.1} \) as \( x \to 0. \)

• EXERCISE 3.6.13
  \( x^{-1} \) and \( -\ln(x) \) as \( x \to 0. \) Use your result to figure out \( \lim_{x \to 0} x \ln(x). \)

• EXERCISE 3.6.14
  \( x^{-1} \) and \( \frac{1}{x^2} \) as \( x \to 0. \)

• EXERCISE 3.6.15
  \( x^2 \) and \( x^3 \) as \( x \to 0. \)

• EXERCISE 3.6.16
  \( x^2 \) and \( e^x - x - 1 \) as \( x \to 0. \)

♠ For each of the following functions, find the leading behavior of the numerator, the denominator, and the whole function at both 0 and ∞. Find the limit of the function at 0 and ∞ (and check with L’Hopital’s rule when appropriate). Use the method of matched leading behaviors to sketch a graph.

• EXERCISE 3.6.17
  \( A_1(c) = \frac{2c^2}{1+c}. \)

• EXERCISE 3.6.18
  \( A_2(c) = \frac{c^2}{1+2c}. \)

• EXERCISE 3.6.19
  \( A_3(c) = \frac{1+c+c^2}{1+c}. \)

• EXERCISE 3.6.20
  \( A_4(c) = \frac{1+c}{1+c+c^2}. \)
EXERCISE 3.6.21
\[ A_5(c) = \frac{3c}{1 + \ln(1 + c)}. \]

EXERCISE 3.6.22
\[ A_6(c) = \frac{(c^c + 1)}{e^{2c} + 1}. \]

Write the tangent line approximation for the numerators and denominators of the following functions and show that the result of applying L'Hopital's rule matches that of comparing the linear approximations.

EXERCISE 3.6.23
\[ f(x) = \frac{2x + x^2}{3x + 2x^2} \text{ at } x = 0. \]

EXERCISE 3.6.24
\[ f(x) = \frac{\ln(1 + x)}{e^{2x} - 1} \text{ at } x = 0. \]

EXERCISE 3.6.25
\[ f(x) = \frac{\ln|x|}{x^2 - 1} \text{ at } x = 1. \]

EXERCISE 3.6.26
\[ f(x) = \frac{\cos(x) + 1}{\sin(x)} \text{ at } x = \pi. \]

APPLICATIONS

Use the method of leading behavior, L'Hopital's rule, and the method of matched leading behaviors to graph the following absorption functions.

EXERCISE 3.6.27
\[ A(c) = \frac{5c}{1 + c}. \]

EXERCISE 3.6.28
\[ A(c) = \frac{c}{1 + c}. \]

EXERCISE 3.6.29
\[ A(c) = \frac{5c^2}{1 + c^2}. \]

EXERCISE 3.6.30
\[ A(c) = \frac{5c}{1 + c}. \]

EXERCISE 3.6.31
\[ A(c) = \frac{5c^2}{1 + c^2}. \]

EXERCISE 3.6.32
\[ A(c) = 5c(1 + c). \]

Use the method of matched leading behaviors to graph the following Hill functions (equation 2.5) and their variants.

EXERCISE 3.6.33
\[ h_3(x) = \frac{x^3}{1 + x^3}. \]

EXERCISE 3.6.34
\[ g_3(x) = \frac{10 + x^3}{x^3}. \]

EXERCISE 3.6.35
\[ h_{10}(x) = \frac{x^{10}}{1 + x^{10}}. \]

EXERCISE 3.6.36
\[ g_{10}(x) = \frac{x^{10}}{0.1 + x^{10}}. \]

The following discrete-time dynamical systems describe the populations of two competing strains of bacteria

\[ a_{t+1} = sa_t, \]
\[ b_{t+1} = rb_t. \]

For the following values of the initial conditions \(a_0\) and \(b_0\), and the per capita reproduction \(s\) and \(r\),

a. Find the number of each type as a function of time,

b. Find the fraction of type \(a\) as a function of time,
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· Use leading behavior or L'Hopital's rule to find the limit of the fraction as $t \to \infty$.

· Compute the fraction at $t = 0, 10, 20$ and $50$ and compare with your limit.

* EXERCISE 3.6.37
  $a_0 = 10^4, b_0 = 10^6, s = 2.0, r = 1.5.$

* EXERCISE 3.6.38
  $a_0 = 10^4, b_0 = 10^6, s = 1.5, r = 2.0.$

* EXERCISE 3.6.39
  $a_0 = 10^4, b_0 = 10^6, s = 0.8, r = 1.2.$

* EXERCISE 3.6.40
  $a_0 = 10^4, b_0 = 10^6, s = 0.5, r = 0.3.$

♠ Many of our absorption equations are of the form

$$A(c) = \frac{r(c)}{\alpha + r(c)}$$

where $\alpha$ and $k$ are positive parameters, and where $r(0) = 0, \lim_{c \to \infty} r(c) = \infty$ and $r'(c) > 0$. In each of the following cases, identify $r(c)$ and show that $A(c)$ is increasing. Use L’Hopital’s rule to find the limit as $c \to \infty$, and use the method of leading behavior to describe absorption near $c = 0$ and $c = \infty$.

* EXERCISE 3.6.41
  $A(c) = \frac{\alpha c}{k + c}$ (from table 3.1).

* EXERCISE 3.6.42
  $A(c) = \frac{\alpha c^2}{k + c^2}$ (from table 3.1).

* EXERCISE 3.6.43
  $A(c) = \frac{\alpha c^n}{k + c^n}$ for some positive value of $n$.

* EXERCISE 3.6.44
  Try without plugging in a particular form for $r(c)$.
Chapter 4

Answers

3.6.1. $f_0(x) = 1$, $f_\infty(x) = x$.

3.6.3. $h_0(z) = e^z$, $h_\infty(z) = e^z$.

3.6.5. $m_0(a) = \frac{1}{a}$, $m_\infty(a) = 30a^2$.

3.6.7. The exponential function $e^{2x}$ approaches infinity faster. By L'Hopital's rule,

$$\lim_{x \to \infty} \frac{e^{2x}}{2^x} = \lim_{x \to \infty} \frac{2e^{2x}}{2^x} = \lim_{x \to \infty} \frac{4e^{2x}}{2} = \infty.$$  

3.6.9. The power function $0.1x^{0.5}$ approaches infinity faster. By L'Hopital's rule,

$$\lim_{x \to \infty} \frac{0.1x^{0.5}}{30 \ln(x)} = \lim_{x \to \infty} \frac{0.05x^{-0.5}}{30x^{-1}} = \lim_{x \to \infty} 0.0017x^{0.5} = \infty.$$  

3.6.11. The exponential function $e^{2x}$ approaches zero faster. L'Hopital's rule doesn't really make things simpler directly, but

$$\lim_{x \to \infty} \frac{e^{-2x}}{x^{-2}} = \lim_{x \to \infty} \frac{x^{2}}{e^{2x}} = \lim_{x \to \infty} \frac{2x}{2e^{2x}} = \lim_{x \to \infty} \frac{2}{4e^{2x}} = 0.$$  

3.6.13. I would guess that the power function approaches infinity faster.

$$\lim_{x \to \infty} \frac{x^{-1}}{-\ln(x)} = \lim_{x \to \infty} \frac{-x^{-2}}{-1} = \lim_{x \to \infty} x^{-1} = \infty.$$  

Therefore, $\lim_{x \to 0} \frac{1}{x \ln(x)} = \infty$, and $\lim_{x \to 0} x \ln(x) = 0$.

3.6.15. The power function with the larger power, $x^3$, approaches 0 more quickly.

$$\lim_{x \to \infty} \frac{x^{3}}{x^{2}} = \lim_{x \to \infty} \frac{3x^{2}}{2x} = \lim_{x \to \infty} \frac{6x}{2} = 0.$$  

3.6.17. The numerator has only one term, so the leading behavior is $2c^2$ at both 0 and $\infty$. The denominator has leading behavior 1 for $c$ near 0 and $c$ for $c$ large. Therefore,

$$A_1(c)_0 = \frac{2c^2}{1} = 2c^2$$

$$A_1(c)_\infty = \frac{2c^2}{c} = 2c$$

$$\lim_{c \to 0} A_1(c) = 0$$

$$\lim_{c \to \infty} A_1(c) = \infty.$$  

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L'Hopital's rule is not appropriate at \( c = 0 \), because the denominator approaches 1. This limit can be found by plugging in. As \( c \to \infty \), both the numerator and denominator approach infinity, so we can use L'Hopital's rule to check

\[
\lim_{c \to \infty} \frac{2c^2}{1 + c} = \lim_{c \to \infty} \frac{4c}{1} = \infty.
\]

**3.6.19.** The numerator has leading behavior 1 near 0 and \( c^2 \) for \( c \) large. The denominator has leading behavior 1 for \( c \) near 0 and \( c \) for \( c \) large. Therefore,

\[
A_3(c)_0 = \frac{1}{1} = 1
\]

\[
A_3(c)_\infty = \frac{c^2}{c} = c
\]

\[
\lim_{c \to 0} A_3(c) = 1
\]

\[
\lim_{c \to \infty} A_3(c) = \infty.
\]

L'Hopital's rule is not appropriate at \( c = 0 \), because both the numerator and denominator approach 1. This limit can be found by plugging in. As \( c \to \infty \), both the numerator and denominator approach infinity, so we can use L'Hopital's rule to check

\[
\lim_{c \to \infty} \frac{1 + c + c^2}{1 + c} = \lim_{c \to \infty} \frac{1 + 2c}{1} = \infty.
\]

**3.6.21.** The numerator has only one term, so the leading behavior is \( 3c \) at both 0 and \( \infty \). The denominator has leading behavior 1 for \( c \) near 0 and \( \ln(1 + c) \) for \( c \) large. Therefore,

\[
A_5(c)_0 = \frac{3c}{1} = 3c
\]

\[
A_5(c)_\infty = \frac{3c}{\ln(1 + c)}
\]

\[
\lim_{c \to 0} A_5(c) = 0
\]

\[
\lim_{c \to \infty} A_5(c) = \infty.
\]

L'Hopital's rule is not appropriate at \( c = 0 \), because the denominator approaches 1. This limit can be found by plugging in. As \( c \to \infty \), both the numerator and denominator approach infinity, so we can use L'Hopital's rule to check

\[
\lim_{c \to \infty} \frac{3c}{1 + \ln(1 + c)} = \lim_{c \to \infty} \frac{3}{1+c} = \lim_{c \to \infty} \frac{3(1+c)}{1} = \infty.
\]
3.6.23. The tangent line to $2x + x^2$ near $x = 0$ is $2x$. The tangent line to $3x + 2x^2$ near $x = 0$ is $3x$. For small $x$, $f(x) \approx \frac{2x}{3x} = 2/3$. With L’Hopital’s rule,
\[
\lim_{x \to 0} \frac{2x + x^2}{3x + 2x^2} = \lim_{x \to 0} \frac{2 + 2x}{3 + 4x} = 2/3.
\]

3.6.25. The tangent line to $\ln(x)$ near $x = 1$ is $x - 1$. The tangent line to $x^2 - 1$ near $x = 1$ is $2(x - 1)$. For small $x$, $f(x) \approx \frac{x - 1}{2(x - 1)} = 1/2$. With L’Hopital’s rule,
\[
\lim_{x \to 1} \frac{\ln(x)}{x^2 - 1} = \lim_{x \to 1} \frac{1/x}{2x} = 1/2.
\]

3.6.27. This function acts like $5c$ for small $c$ and like $5$ for large $c$.

3.6.29. This function acts like $5c^2$ for small $c$ and like $5$ for large $c$.

3.6.31. This function acts like $5c$ for small $c$ and decreases to $0$ like $5/c$ for large $c$. 
3.6.33. \( h_3(x)_0 = x^3, h_3(x)_\infty = 1. \)

3.6.35. \( h_{10}(x)_0 = x^{10}, h_{10}(x)_\infty = 1. \)

3.6.37.

a. \( a_t = 10^4 \cdot 2.0^t, b_t = 10^6 \cdot 1.5^t. \)

b. \( p_t = \frac{10^4 \cdot 2.0^t}{10^4 \cdot 2.0^t + 10^6 \cdot 1.5^t}. \)

c. In exponential notation, we can rewrite the denominator as \( 10^4 e^{\ln(2.0)t} + 10^6 e^{\ln(1.5)t} \), with leading behavior \( 10^4 e^{\ln(2.0)t} \) because the parameter in the exponent is larger. Therefore, \( \lim_{t \to 0} p_t = 1. \)

d. \( p_0 = 0.01, p_{10} = 0.15, p_{20} = 0.76, p_{50} = 0.9999. \) This is mightly close to the limit.

3.6.39.

a. \( a_t = 10^4 \cdot 0.8^t, b_t = 10^5 \cdot 1.2^t. \)

b. \( p_t = \frac{10^4 \cdot 0.8^t}{10^4 \cdot 0.8^t + 10^5 \cdot 1.2^t}. \)

c. In exponential notation, the denominator is \( 10^4 e^{\ln(0.8)t} + 10^5 e^{\ln(1.2)t} = 10^4 e^{-0.223t} + 10^5 e^{0.182t}. \) The leading behavior is \( 10^4 e^{0.182t} \) because this term grows and the other term shrinks. Therefore, \( \lim_{t \to 0} p_t = 0. \)

d. \( p_0 = 0.09, p_{10} = 0.0017, p_{20} = 0.00003, p_{50} = 1.5 \times 10^{-10}. \) This is extremely close to the limit.

3.6.41. \( r(c) = c, \) the derivative is \( A'(c) = \frac{\alpha k}{(k + c)^2} > 0. \) By L’Hospital’s rule,

\[
\lim_{c \to \infty} \alpha \frac{c}{k + c} = \lim_{c \to \infty} \alpha \frac{1}{1} = \alpha.
\]

Near \( c = 0, \) the leading behavior of the denominator is \( k, \) so \( A(c) \approx \frac{\alpha}{k} \). For large \( c, \) the leading behavior of the denominator is \( c, \) so \( A(c) \approx \alpha. \)

3.6.43. \( r(c) = c^n, \) the derivative is \( A'(c) = \frac{n \alpha k c^{n-1}}{(k + c)^2} > 0. \) By L’Hospital’s rule,

\[
\lim_{c \to \infty} \alpha \frac{c^n}{k + c^2} = \lim_{c \to \infty} \alpha \frac{n c^{n-1}}{n c^{n-1}} = \alpha.
\]

Near \( c = 0, \) the leading behavior of the denominator is \( k, \) so \( A(c) \approx \frac{\alpha}{k} \). For large \( c, \) the leading behavior of the denominator is \( c^n, \) so \( A(c) \approx \alpha. \)