2.7 The Second Derivative, Curvature, and Acceleration

MATHEMATICAL TECHNIQUES

On the figures, label

a. One critical point.
b. One point with a positive derivative.
c. One point with a negative derivative.
d. One point with a positive second derivative.
e. One point with a negative second derivative.
f. One point of inflection.

• EXERCISE 2.7.1

• EXERCISE 2.7.2

• EXERCISE 2.7.3

• EXERCISE 2.7.4
Draw graphs of functions with the following properties.

- **EXERCISE 2.7.5**
  A function with a positive, increasing derivative.
- **EXERCISE 2.7.6**
  A function with a positive, decreasing derivative.
- **EXERCISE 2.7.7**
  A function with a negative, increasing (becoming less negative) derivative.
- **EXERCISE 2.7.8**
  A function with a negative, decreasing (becoming more negative) derivative.

Although there is no easy way to recognize all points with positive or negative third derivative, it is possible for some points (usually points of inflection).

- **EXERCISE 2.7.9**
  On the figure for exercise 2.7.1, find one point with negative third derivative.
- **EXERCISE 2.7.10**
  On the figure for exercise 2.7.2, find one point with positive third derivative.

Find the first and second derivatives of the following functions.

- **EXERCISE 2.7.11**
  \( s(x) = 1 - x + x^2 - x^3 + x^4. \)
- **EXERCISE 2.7.12**
  \( g(z) = 3z^3 + 2z^2. \)
- **EXERCISE 2.7.13**
  \( h(y) = y^{10} - y^9. \)
- **EXERCISE 2.7.14**
  \( p(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}. \)
- **EXERCISE 2.7.15**
  \( F(z) = z(1 + z)(2 + z). \)
- **EXERCISE 2.7.16**
  \( R(s) = (1 + s^2)(2 + s). \)
- **EXERCISE 2.7.17**
  \( f(x) = \frac{3 + x}{2x}. \)
- **EXERCISE 2.7.18**
  \( G(y) = \frac{2 + y}{y^2}. \)

Find the first and second derivatives of the following functions and use them to sketch a graph.

- **EXERCISE 2.7.19**
  \( f(x) = x^{-3} \) for \( x > 0. \)
- **EXERCISE 2.7.20**
  \( g(z) = z + \frac{1}{z} \) for \( z > 0. \)
- **EXERCISE 2.7.21**
  \( h(x) = (1 - x)(2 - x)(3 - x). \)
- **EXERCISE 2.7.22**
  \( M(t) = \frac{1}{t^2} \) for \( t > 0. \)
- **EXERCISE 2.7.23**
  \( f(x) = 2x^3 + 1 \) for \(-5 \leq x \leq 5. \)
2.7. THE SECOND DERIVATIVE, CURVATURE, AND ACCELERATION

- **EXERCISE 2.7.24**
  \[ f(x) = \frac{1}{x^2} \text{ for } 0 < x \leq 2. \]

- **EXERCISE 2.7.25**
  \[ f(x) = 10x^2 - 50x \text{ on } -5 \leq x \leq 5. \]

- **EXERCISE 2.7.26**
  \[ f(x) = x - x^2 \text{ on } 0 \leq x \leq 1. \]

Some higher derivatives can be found without a lot of calculation.

- **EXERCISE 2.7.27**
  Find the 10th derivative of \( x^5 \).

- **EXERCISE 2.7.28**
  Find the first 6 derivatives of \( x^5 \).

- **EXERCISE 2.7.29**
  Is the 8th derivative of \( p(x) = 7x^8 - 8x^7 - 5x^6 + 6x^5 - 4x^3 \) positive or negative?

- **EXERCISE 2.7.30**
  Find the 5th derivative of \( x(1 + x)(2 + x)(3 + x)(4 + x) \).

We can approximate a function with the tangent line, which matches the value of the function and its first derivative. A better approximation uses a parabola that matches the value of the function, and its first and second derivatives. For each of the following functions,

- Find the tangent line at \( x = 1 \).
- Add a quadratic term to the formula of your tangent line to match the second derivative at \( x = 1 \).
- Sketch a graph of the function, its tangent line, and the approximating quadratic for \( 0.5 < x < 1.5 \).

- **EXERCISE 2.7.31**
  \[ f(x) = x^{-3}. \]

- **EXERCISE 2.7.32**
  \[ g(x) = x + \frac{1}{x}. \]

**APPLICATIONS**

The following equations give the positions as functions of time of objects tossed from towers in various exotic solar system locations. For each,

- Find the velocity and the acceleration of this object.
- Sketch a graph of the position for \( 0 \leq t \leq 3 \).
- How high was the tower? Which way was the object thrown? How does the acceleration compare with that on earth (9.8 m/sec^2)?

- **EXERCISE 2.7.33**
  An object on Saturn that follows \( p(t) = -5.2t^2 - 2.0t + 50.0 \).

- **EXERCISE 2.7.34**
  An object on the Sun that follows \( p(t) = -137t^2 + 20.0t + 500.0 \).

- **EXERCISE 2.7.35**
  An object on Pluto that follows \( p(t) = -0.325t^2 - 20.0t + 500.0 \).

- **EXERCISE 2.7.36**
  An object on Mercury that follows \( p(t) = -1.85t^2 + 20.0t \).

The total mass is the product of the following functions for mass and number as functions of time in years (exercises 2.6.23–2.6.26). Find the second derivative of each and check your graph.

- **EXERCISE 2.7.37**
  (based on exercise 2.6.23) The population \( P \) is \( P(t) = 2.0 \times 10^6 + 2.0 \times 10^4t \) and the weight per person \( W(t) \) is \( W(t) = 80 - 0.5t \).
• **EXERCISE 2.7.38**
  (based on exercise 2.6.24) The population $P$ is $P(t) = 2.0 \times 10^6 - 2.0 \times 10^4 t$ and the weight per person $W(t)$ is $W(t) = 80 + 0.5t$.

• **EXERCISE 2.7.39**
  (based on exercise 2.6.25) The population $P$ is $P(t) = 2.0 \times 10^6 + 1000t^2$ and the weight per person $W(t)$ is $W(t) = 80 - 0.5t$.

• **EXERCISE 2.7.40**
  (based on exercise 2.6.26) The population $P$ is $P(t) = 2.0 \times 10^6 + 2.0 \times 10^4 t$ and the weight per person $W(t)$ is $W(t) = 80 - 0.005t^2$.

In a model of a growing population, we find the new population by multiplying the old population by the per capita reproduction. For each case, find the second derivative of the new population as a function of the old population and sketch a graph.

• **EXERCISE 2.7.41**
  Per capita reproduction is $2.0(1 - \frac{b_t}{1000})$. Consider values of $b_t$ less than 1000.

• **EXERCISE 2.7.42**
  Per capita reproduction is $2.0b_t(1 - \frac{b_t}{1000})$. Consider values of $b_t$ less than 1000.
Chapter 4

Answers

2.7.1.

2.7.3.

2.7.5.

2.7.7.
2.7.9. The point of inflection marked F has a negative third derivative because the second derivative changes from positive to negative values and is therefore decreasing.

2.7.11. \( s'(x) = -1 + 2x - 3x^2 + 4x^3. \) \( s''(x) = 2 - 6x + 12x^2. \)

2.7.13. \( h'(y) = 10y^9 - 9y^8. \) \( h''(y) = 90y^8 - 72y^7. \)

2.7.15. \( F'(z) = (1+z)(2+z) + z(2+z) + z(1+z). \) \( F''(z) = (1+z) + (2+z) + z + (2+z) + z + (1+z) = 6z + 6. \)

2.7.17. \( f'(x) = \frac{28 - 2(3 + x)}{4x^2} = \frac{-3}{2x^2}. \) \( f''(x) = -\frac{3}{x^3}. \)

2.7.19. \( f'(x) = -3x^{-4} < 0 \) so the function is decreasing, \( f''(x) = 12x^{-5} > 0 \) so the function is always concave up.

2.7.21. \( h'(x) = -3x^2 + 12x - 11, \) which has solutions at 2.577 and 1.422. \( h''(x) = -6x + 12 \) so the function is concave up for \( x < 0.5 \) and concave down for \( x > 0.5. \)

2.7.23. \( f'(x) = 6x^2. \) This is always positive. \( f''(x) = 12x. \) This is positive when \( x > 0. \)

2.7.25. \( f'(x) = 20x - 50. \) This is positive when \( x > 2.5 \) and negative when \( x < 2.5. \) \( f''(x) = 20. \) This is always positive.
2.7.27. It is 0, because the degree will have been reduced to 0.

2.7.29. Positive.

2.7.31. The derivative is $f'(x) = -3x^{-4}$ and the second derivative is $f''(x) = 12x^{-15}$.

a. Because $f(1) = 1$ and $f'(1) = -3$, the tangent line is $\hat{f}(x) = 1 - 3(x - 1)$.

b. Because $f''(1) = 12$, the quadratic $\hat{f}(x) = 1 - 3(x - 1) + 6(x - 1)^2$ matches both the first and second derivatives at $x = 1$.

c. 

2.7.33.

a. Velocity is $p'(t) = -10.4t - 2.0$. Acceleration is $p''(t) = -10.4$.

b. The position at $t = 3$ is less than 0, so the object has already hit the ground.

c. The tower was 50.0 meters high, and the object was thrown downward at 2.0 m/sec. The acceleration of gravity on Saturn is only slightly greater than that on earth, probably due to Saturn’s low density.

2.7.35.

a. Velocity is $p'(t) = -0.65t - 20.0$. Acceleration is $p''(t) = -0.65$. 
c. The tower was 500.0 meters high, and the object was thrown downward at 20.0 m/sec. The acceleration of gravity on Pluto is tiny, so the object is falling only slightly faster after 3 seconds than it was when it was thrown.

2.7.37. The second derivative is -20000, matching the graph which is always concave down.

2.7.39. The second derivative is $160000 - 3000t$, which is positive for $t \leq 53.33$ and negative thereafter. This matches a graph which switches from being concave up to concave down.

2.7.41. The derivative of $f(b) = 2.0b(1 - \frac{b}{1000})$ is $f'(b) = 2.0 - \frac{4.0b}{1000}$, and the second derivative is $-\frac{4.0}{1000}$ which is always negative. The graph is always concave down.