2.6 Derivatives of products and quotients

MATHEMATICAL TECHNIQUES

♦ Find the derivatives of the following functions using the product rule.
  • EXERCISE 2.6.1
    \[ f(x) = (2x + 3)(-3x + 2). \]
  • EXERCISE 2.6.2
    \[ g(z) = (5z - 3)(z + 2). \]
  • EXERCISE 2.6.3
    \[ r(y) = (5y - 3)(y^2 - 1). \]
  • EXERCISE 2.6.4
    \[ s(t) = (t^2 + 2)(3t^2 - 1) \]
  • EXERCISE 2.6.5
    \[ h(x) = (x + 2)(2x + 3)(-3x + 2) \text{ (apply the product rule twice).} \]
  • EXERCISE 2.6.6
    \[ F(w) = (w - 1)(2w - 1)(3w - 1) \text{ (apply the product rule twice).} \]

♦ Find the derivatives of the following functions using the quotient rule.
  • EXERCISE 2.6.7
    \[ f(x) = \frac{1 + x}{2 + x}. \]
  • EXERCISE 2.6.8
    \[ f(x) = \frac{x^2}{1 + 2x^3}. \]
  • EXERCISE 2.6.9
    \[ g(z) = \frac{1 + z^2}{1 + 2z^5}. \]
  • EXERCISE 2.6.10
    \[ h(z) = \frac{1 + 2z^3}{(1 + z^2)^2}. \]
  • EXERCISE 2.6.11
    \[ F(x) = \frac{1 + x}{(2 + x)(3 + x)}. \]
  • EXERCISE 2.6.12
    \[ G(x) = \frac{(1 + x)(2 + x)}{3 + x}. \]

♦ For the following functions, use base point \( x_0 = 1.0 \) and \( \Delta x = 0.1 \) to compute \( \Delta f \) and \( \Delta g \). Find \( \Delta fg \) (the change in the product) by computing \( f(x_0 + \Delta x)g(x_0 + \Delta x) - f(x_0)g(x_0) \). Check that \( \Delta fg = g(x_0)\Delta f + f(x_0)\Delta g + \Delta f\Delta g \). Try the same with \( \Delta x = 0.01 \) and see whether the term \( \Delta f\Delta g \) becomes very small.
  • EXERCISE 2.6.13
    \[ f(x) = 2x + 3 \text{ and } g(x) = -3x + 2. \]
  • EXERCISE 2.6.14
    \[ f(x) = x^2 + 2 \text{ and } g(x) = 3x^2 - 1. \]

♦ Suppose \( p(x) = f(x)g(x) \). Test out the incorrect formula \( p'(x) = f'(x)g'(x) \) on the following functions.
  • EXERCISE 2.6.15
    \[ f(x) = x, g(x) = x^2. \]
  • EXERCISE 2.6.16
    \[ f(x) = 1, g(x) = x^3. \]

♦ For positive integer powers, it is possible to derive the power rule with mathematical induction. The idea is to show that a formula is true for \( n = 1 \), and then that if it is true for some particular \( n \), it must then also be true for \( n + 1 \).
  • EXERCISE 2.6.17
    Check that the power rule is true for \( n = 1 \).
  • EXERCISE 2.6.18
    Use the product rule on \( x^2 = x \cdot x \) to check the power rule for \( n = 2 \) using only the power rule with \( n = 1 \).
2.6. DERIVATIVES OF PRODUCTS AND QUOTIENTS

• **EXERCISE 2.6.19**  
  Use the product rule on \( x^3 = x^2 \cdot x \) to check the power rule for \( n = 3 \) using only the power rule with \( n = 1 \) and \( n = 2 \).

• **EXERCISE 2.6.20**  
  Assuming that the power rule is true for \( n \), find \( \frac{d(x^{n+1})}{dx} \) using the product rule, and check that it too satisfies the power rule.

**APPLICATIONS**

♣ Find the derivative of the updating function from equation 1.51, \( f(p) = \frac{sp}{sp + r(1 - p)} \), with the following values of the parameters \( s \) and \( r \).

• **EXERCISE 2.6.21**  
  \( s = 1.2, r = 2.0. \)

• **EXERCISE 2.6.22**  
  \( s = 1.8, r = 0.8. \)

♣ The total mass of the population is the product of the number of individuals and the mass of each individual. In each case, time is measured in years and mass is measured in kilograms.

  a. Find the total mass as a function of time,

  b. Compute the derivative,

  c. Find the population, the mass of each individual, and the total mass at the time when the derivative is equal to zero.

  d. Sketch a graph of the total mass over the next 100 years,

• **EXERCISE 2.6.23**  
  The population \( P \) is \( P(t) = 2.0 \times 10^6 + 2.0 \times 10^4 t \) and the weight per person \( W(t) \) is \( W(t) = 80 - 0.5 t \).

• **EXERCISE 2.6.24**  
  The population \( P \) is \( P(t) = 2.0 \times 10^6 - 2.0 \times 10^4 t \) and the weight per person \( W(t) \) is \( W(t) = 80 + 0.5 t \).

• **EXERCISE 2.6.25**  
  The population \( P \) is \( P(t) = 2.0 \times 10^6 + 1000 t^2 \) and the weight per person \( W(t) \) is \( W(t) = 80 - 0.5 t \).

• **EXERCISE 2.6.26**  
  The population \( P \) is \( P(t) = 2.0 \times 10^6 + 2.0 \times 10^4 t \) and the weight per person \( W(t) \) is \( W(t) = 80 - 0.005 t^2 \).

♣ In each of the following situations (based on exercise 2.5.38), the mass is the product of the density and the volume. In each case, time is measured in days and density is measured in grams per cm\(^3\).

  a. Find the mass as a function of time.

  b. Compute the derivative.

  c. Sketch a graph of the mass over the next 30 days.

• **EXERCISE 2.6.27**  
  The above-ground volume is \( V_a(t) = 3.0 t + 20.0 \) and the above-ground density is \( \rho_a(t) = 1.2 - 0.01 t \).

• **EXERCISE 2.6.28**  
  The below-ground volume is \( V_b(t) = -1.0 t + 40.0 \) and the below-ground density is \( \rho_b(t) = 1.8 + 0.02 t \).

♣ Suppose that the fraction of chicks that survive, \( P(N) \), as a function of the number \( N \) of eggs laid is given by the following forms (variants of the model studied in section 2.5). The total number of offspring that survive is \( S(N) = NP(N) \). Find the number of surviving offspring when the bird lays 1, 5, or 10 eggs. Find \( S'(N) \). Sketch a graph of \( S(N) \). What do you think is the best strategy for each bird?

• **EXERCISE 2.6.29**  
  \( P(N) = 1 - 0.08 N. \)

• **EXERCISE 2.6.30**  
  \( P(N) = 1 - 0.16 N. \)
• **EXERCISE 2.6.31**
  \[ P(N) = \frac{1}{1 + 0.9N}. \]

• **EXERCISE 2.6.32**
  \[ P(N) = \frac{1}{1 + 0.1N^2}. \]

♠ Suppose that the mass \( M(t) \) of an insect (in grams) and the volume \( V(t) \) (in cm\(^3\)) are known functions of time (in days).

  a. Find the density \( \rho(t) \) as a function of time.
  
  b. Find the derivative of the density.
  
  c. At what times is the density increasing?
  
  d. Sketch a graph of the density over the first 5 days.

• **EXERCISE 2.6.33**
  \[ M(t) = 1 + t^2 \text{ and } V(t) = 1 + t. \]

• **EXERCISE 2.6.34**
  \[ M(t) = 1 + t^2 \text{ and } V(t) = 1 + 2t. \]

♠ In a discrete-time dynamical system describing the growth of a population in the absence of immigration and emigration, the new population is the product of the old population and the per capita reproduction. Represent the old population by \( b_t \). In each case, find the new population as a function of the old population, find the derivative, and sketch the function.

• **EXERCISE 2.6.35**
  Per capita reproduction is \( 2.0(1 - \frac{b_t}{1000}) \). Consider values of \( b_t \) less than 1000.

• **EXERCISE 2.6.36**
  Per capita reproduction is \( \frac{2.0}{1 + \frac{b_t}{1000}} \). Consider values of \( b_t \) less than 2000.

♠ The following steps should help you to figure out what happens to the Hill function \( h_n(x) = \frac{x^n}{1 + x^n} \) for large values of \( n \).

  a. Compute the value of the function at \( x = 0, x = 1 \) and \( x = 2 \).
  
  b. Compute the derivative and evaluate at \( x = 0, x = 1 \) and \( x = 2 \).
  
  c. Sketch a graph.
  
  d. \( h_n(x) \) can be thought of as representing a response to a stimulus of strength \( x \). Would the response work as a good switch, giving a small output for inputs less than 1 and a large output for inputs greater than 1?

• **EXERCISE 2.6.37**
  With \( n = 3 \).

• **EXERCISE 2.6.38**
  With \( n = 10 \).

Chapter 4

Answers

2.6.1. \( f'(x) = 2 \cdot (-3x + 2) + (3) \cdot (2x + 3) = -12x - 5 \).

2.6.3. \( r'(y) = 5(y^2 - 1) + 2y(5y - 3) \).

2.6.5.
\[
\frac{dh}{dx} = \frac{d(x + 2)}{dx} \left( -3x + 2 \right) + \frac{d(2x + 3)}{dx} \left( -12x - 5 \right)
= 1 \cdot (2x + 3)(-3x + 2) + (x + 2) \cdot (-12x - 5)
= (2x + 3)(-3x + 2) - (x + 2)(12x + 5).
\]

2.6.7. Set \( u(x) = 1 + x \) and \( v(x) = 2 + x \). Then \( u'(x) = 1 \) and \( v'(x) = 2 \). By the quotient rule,
\[
f'(x) = \frac{(2 + x) \cdot 1 - (1 + x) \cdot 2}{(2 + x)^2} = \frac{1}{(2 + x)^2}.
\]

2.6.9. \( g'(z) = \frac{-2z(-1+3x+z^3)}{(1+3x)^2} \).

2.6.10. \( h'(z) = \frac{2z(-1+3x+z^3)}{(1+3x)^2} \).

2.6.11. Set \( u(x) = 1 + x \) and \( v(x) = (2 + x)(3 + x) \). Then \( u'(x) = 1 \) and \( v'(x) = (2 + x) + (3 + x) = 5 + 2x \), by the product rule.
\[
F'(x) = \frac{(2 + x)(3 + x) \cdot 1 - (1 + x)(5 + 2x)}{(2 + x)^2(3 + x)^2} = \frac{1 - 2x - x^2}{(2 + x)^2(3 + x)^2}
\]

2.6.13. \( \Delta f = 0.2, \Delta g = -0.3, \) and \( \Delta f g = -1.76 \). This is equal to \( g(x_0)\Delta f + f(x_0)\Delta g + \Delta f \Delta g = -0.2 - 1.5 - 0.06 \). With \( \Delta x = 0.01, \Delta f = 0.02, \Delta g = -0.03, \) and \( \Delta f g = -0.1706 \). This is equal to \( g(x_0)\Delta f + f(x_0)\Delta g + \Delta f \Delta g = -0.02 - 0.15 - 0.0006 \). The last term is now much smaller than the rest.

2.6.15. Would get that the derivative of \( x^3 \) is \( 2x \), which is wrong.

2.6.17. The power rule works for \( n = 1 \) because \( x^1 = x \) is a linear function with derivative \( 1 = 1x^0 \).

2.6.19. \( \frac{dx^3}{dx} = x^2 \cdot \frac{dx}{dx} + \frac{dx}{dx} = x^2 + 2x^2 = 3x^2 \). It worked.

2.6.21. This simplifies to \( \frac{2.4}{(1.2p + 2.0(1 - p))^2} \).

2.6.23. We denote the total mass by \( T(t) \).

a. \( T(t) = P(t)W(T) = (2.0 \times 10^6 + 2.0 \times 10^4 t)(80 - 0.5t) \).

b. \( T'(t) = 6.0 \times 10^6 - 2.0 \times 10^4 t \).

c. \( T'(t) = 0 \) when \( 6.0 \times 10^5 - 2.0 \times 10^4 t = 0 \) or when \( t = 30 \text{ yr} \). At time 30, the population is \( 2.6 \times 10^6 \), the weight per person is 65 kg, and the total weight is \( 1.69 \times 10^8 \text{ kg} \).
2.6.25. We denote the total mass by $T(t)$.

a. $T(t) = P(t)W(T) = (2.0 \times 10^6 + 1000t^2)(80 - 0.5t)$.

b. $T'(t) = 1.0 \times 10^6 + 1600000t - 1500.00t^2$.

c. $T'(t) = 0$ at $t = 6.67$ and $t = 100.0$. At time 6.67, the population is $2.04 \times 10^6$, the weight per person is 76.67 kg, and the total weight is $1.56 \times 10^8$ kg. At time 100, the population is $1.2 \times 10^7$, the weight per person is 30.0 kg, and the total weight is $3.6 \times 10^8$ kg.

2.6.27. Total mass above ground is $(3.0t + 20)(1.2 - 0.01t)$. The derivative is $-0.06t + 3.4$. This switches from positive to negative at $t = 56.7$, and thus the above-ground mass increases during the entire first 30 days.

2.6.29. $S(1) = 0.92$, $S(5) = 3.0$, $S(10) = 2.0$. $S'(N) = 1 - 0.16N$. This bird does best by laying about 6 eggs.
2.6.31. $S(1) = 0.67, S(5) = 1.43, S(10) = 1.67$. $S'(N) = \frac{1}{(1+0.5N)^2}$ which is always positive. This bird does best by laying as many eggs as it can.

![Graph](image)

2.6.33.

a. $\rho(t) = \frac{(1 + t^2)}{(1 + t)}$.

b. $\rho'(t) = \frac{(t^2 + 2t - 1)}{(1 + t)^2}$.

c. This is positive when $t^2 + 2t - 1 > 0$. This occurs for $t$ larger than the solution of $t^2 + 2t - 1 = 0$, which can be found with the quadratic formula to be 0.414. After that, the density increases.

![Graph](image)

2.6.35. The new population is the product of the per capita reproduction and the old population $b_t$, or $b_{t+1} = 2.0(1 - \frac{b_t}{1000})b_t$. The derivative of $f(b) = 2.0b(1 - \frac{b}{1000})$ is $f'(b) = 2.0 - \frac{4.0b}{1000}$, which is positive for $b < 500$ and negative for $b > 500$.

![Graph](image)

2.6.37.

a. $h_3(0) = 0, h_3(1) = 0.5, h_3(2) = 0.89$.

b. $h_3'(x) = \frac{3x^2}{(1 + x^3)^2}$. $h_3'(0) = 0, h_3'(1) = 0.75, h_3'(2) = 0.148$.

d. This is kind of a mushy response.