2.3 Continuity

MATHEMATICAL TECHNIQUES

\[ f(x) = \frac{e^x}{x + 1}. \]
\[ h(y) = y^2 \ln(y - 1) \text{ for } y > 1. \]
\[ g(z) = \frac{\ln(z - 1)}{z^2} \text{ for } z > 1. \]
\[ F(t) = \cos(e^t). \]
\[ a(t) = t^2 \text{ if } t > 0 \text{ and } 0 \text{ otherwise.} \]
\[ r(w) = (1 - w)^{-4}. \]
\[ q(z) = (1 + z + z^2)^{-2}. \]

Using the given functions, find the limits by plugging in (if possible). Say whether the limit is infinity or negative infinity. Compute the value of the function 0.1 and 0.01 above and below the limiting argument to see if your answer is correct.

\[ \lim_{t \to 5} l(t) \] (based on exercise 2.3.1).
\[ \lim_{x \to 2} p(x) \] (based on exercise 2.3.2).
\[ \lim_{x \to 0} f(x) \] (based on exercise 2.3.3).
\[ \lim_{y \to 1} h(y) \] (based on exercise 2.3.4).
\[ \lim_{x \to 2} g(z) \] (based on exercise 2.3.5).
\[ \lim_{y \to 2} h(y) \] (based on exercise 2.3.4).
\[ \lim_{x \to 0} g(z) \] (based on exercise 2.3.5).
\[ \lim_{t \to 2} F(t) \] (based on exercise 2.3.6).
\[ \lim_{w \to 1} r(w) \] (based on exercise 2.3.9).
\[ \lim_{t \to 0} a(t) \] (based on exercise 2.3.8).

For the following functions, find the input tolerance necessary to achieve the given output tolerance. Sketch a graph of the function and the tolerance.
• EXERCISE 2.3.21
  How close must the input be to \( x = 0 \) for \( f(x) = x + 2 \) to be within 0.1 of 2?

• EXERCISE 2.3.22
  How close must the input be to \( x = 1 \) for \( f(x) = 2x + 1 \) to be within 0.1 of 3?

• EXERCISE 2.3.23
  How close must the input be to \( x = 1 \) for \( f(x) = x^2 \) to be within 0.1 of 1?

• EXERCISE 2.3.24
  How close must the input be to \( x = 2 \) for \( f(x) = 5x^2 \) to be within 0.1 of 20?

• EXERCISE 2.3.25
  Consider the Heaviside function, defined by
  \[
  \begin{cases} 
  H(x) = 0 & \text{if } x < 0 \\
  H(x) = 1 & \text{if } x \geq 0 
  \end{cases}
  \]
  How close must the input be to \( x = 1 \) for \( H(x) \) to be within 0.1 of 1?

• EXERCISE 2.3.26
  How close must the input be to \( x = 0 \) for the Heaviside function \( H(x) \) (exercise 2.3.25) to be within 0.1 of 0?

  ♦ We can build different continuous approximations of signum (the function giving the sign of a number) as follows. For each, case

  a. Graph the continuous function.
  b. Find the formula.
  c. How close would the input have to be to 0 for the output to be within 0.1 of 0?

• EXERCISE 2.3.27
  A continuous function that is -1 for \( x \leq -0.1 \), 1 for \( x \geq 0.1 \), and is linear for \(-0.1 < x < 0.1\).

• EXERCISE 2.3.28
  A continuous function that is -1 for \( x \leq -0.01 \), 1 for \( x \geq 0.01 \), and is linear for \(-0.01 < x < 0.01\).

APPLICATIONS

♦ Find the accuracy of input necessary to achieve the desired output accuracy.

• EXERCISE 2.3.29
  Suppose the mass of an object as a function of volume is given by \( M = \rho V \). If \( \rho = 2.0 \text{ g/cm}^3 \), how close must \( V \) be to 2.5 cm\(^3\) for \( M \) to be within 0.2 g of 5.0 g?

• EXERCISE 2.3.30
  The area of a disk as a function of radius is given by \( A = \pi r^2 \). How close must \( r \) be to 2.0 cm to guarantee an area within 0.5 cm\(^2\) of 4\(\pi\)?

• EXERCISE 2.3.31
  The flow rate \( F \) through a vessel is proportional to the fourth power of the radius, or
  \[
  F(r) = ar^4 
  \]
  Suppose \( a = 1.0 \text{ cm/sec} \). How close must \( r \) be to 1.0 cm to guarantee a flow within 5\% of 1 ml/sec?

• EXERCISE 2.3.32
  Consider an organism growing according to \( S(t) = S(0)e^{at} \). Suppose \( a = 0.001 \text{ /sec} \), and \( S(0) = 1.0 \text{ mm} \). At time 1000 seconds, \( S(t) = 2.71828 \text{ mm} \). How close must \( t \) be to 1000 seconds to guarantee a size within 0.1 mm of 2.71828 mm?

♦ Suppose a population of bacteria follows the discrete-time dynamical system
  \[
  b_{t+1} = 2.0b_t 
  \]
  and we wish to have a population within \( 1.0 \times 10^8 \) of \( 1.0 \times 10^9 \) at \( t = 10 \).
2.3. **CONTINUITY**

- **EXERCISE 2.3.33**
  What values of $b_0$ produce a result within the desired tolerance? What is the input tolerance?
- **EXERCISE 2.3.34**
  What values of $b_0$ produce a result within the desired tolerance? What is the input tolerance? Why is it harder to hit the target from here?
- **EXERCISE 2.3.35**
  What values of $b_0$ produce a result within the desired tolerance? What is the input tolerance?
- **EXERCISE 2.3.36**
  How would your answers differ if the discrete-time dynamical system were $b_{t+1} = 5b_t$? Would the tolerances be larger or smaller? Why?

* Suppose the amount of toxin in a culture declines according to $T_{t+1} = 0.5T_t$ and we wish to have a concentration within 0.02 of 0.5 g/L at $t = 10$.
- **EXERCISE 2.3.37**
  What values of $T_0$ produce a result within the desired tolerance? What is the input tolerance?
- **EXERCISE 2.3.38**
  What values of $T_0$ produce a result within the desired tolerance? What is the input tolerance?
- **EXERCISE 2.3.39**
  What values of $T_0$ produce a result within the desired tolerance? What is the input tolerance?
- **EXERCISE 2.3.40**
  How would your answers differ if the discrete-time dynamical system were $T_{t+1} = 0.1T_t$? Would the tolerances be larger or smaller? Why?

* Suppose a neuron has the following response to inputs. If it receives a voltage input $V$ greater than or equal to a threshold of $V_0$, it outputs a voltage of $kV$ for some constant $k$. If it receives an input less than the threshold value of $V_0$, it outputs a fixed voltage $V^*$.  
- **EXERCISE 2.3.41**
  Suppose that $k = 2.0$, $V_0 = 50$ and $V^* = 80$. Write and graph the function giving output in terms of input as a function defined in pieces.
- **EXERCISE 2.3.42**
  Suppose that $k = 1.5$, $V_0 = 60$ and $V^* = 100$. Write and graph the function giving output in terms of input as a function defined in pieces.
- **EXERCISE 2.3.43**
  If $k = 2.0$ and $V_0 = 50$, what would $V^*$ have to be for the function to be continuous? Graph the resulting function.
- **EXERCISE 2.3.44**
  If $V_0 = 50$ and $V^* = 80$, what would $k$ have to be to make the function continuous? Graph the resulting function.

* The following questions are based on examples of hysteresis involving children.
- **EXERCISE 2.3.45**
  A child outside is swinging on a swing that makes a horrible screeching noise. Starting from when the swing is furthest back, the pitch of the screeching noise increases as it swings forward and then decreases as it swings back.
  
  a. Draw a graph of the pitch as a function of position without hysteresis.
  b. Draw a graph with hysteresis. Which graph seems more likely?
  c. Imagine what each sounds like. Which is more irritating?

- **EXERCISE 2.3.46**
  Little Billy walks due east to school, but must cross from the south side to the north side of the street. Because he is a very careful child, he crosses at great speed at the first possible opportunity.
  
  a. Graph little Billy’s latitude as a function of distance from home on the way to school.
  b. Graph little Billy’s latitude as a function of distance from home on the way home.
  c. Is this an example of hysteresis?
Chapter 4

Answers

2.3.1. This is a linear function and is continuous everywhere.
2.3.3. This is constructed as the quotient of the continuous exponential function and a continuous linear function. It is guaranteed to be continuous everywhere that the denominator is not equal to 0. The only potential trouble point is \( x = -1 \).
2.3.5. This is a composition of the natural log with a linear function divided by a polynomial. The theorems guarantee continuity except at \( z = 1 \), where we are taking the natural log of 0. At \( z = 0 \), where the denominator is 0, the logarithm is not defined.
2.3.7. This is a composition of the continuous cosine function with a continuous linear function \( (x - \frac{3}{2}) \), and is continuous everywhere.
2.3.9. This is the quotient of the constant 1 by the function \( (1 + w)^4 \), which is a polynomial. This is guaranteed to be continuous except where the denominator is 0, or when \( w = -1 \).
2.3.11. \( f(5) = 31, f(5.1) = 31.5, f(5.01) = 31.05, f(4.9) = 30.5, f(4.99) = 30.95 \).
2.3.13. \( f(0) = e^0/0 + 1 = 1, f(0.1) = 1.0047, f(0.01) = 1.00005, f(-0.1) = 1.0054 \) and \( f(-0.01) = 1.00005 \).
2.3.15. \( g(2) = 0, g(2.1) = 0.022, g(2.01) = 0.0024, g(1.9) = -0.029, g(1.99) = -0.0025 \).
2.3.17. \( g(0) \) cannot be computed. In fact, \( g(z) \) does not make sense for \( z \leq 1 \).
2.3.19. \( r(1) \) cannot be computed. \( r(1.1) = 1.0\times10^4, r(1.01) = 1.0\times10^6, r(0.9) = 1.0\times10^4, r(0.99) = 1.0\times10^6 \). This limit is infinity.
2.3.21. \( f(x) = 2.1 \) if \( x = 0.1 \) and \( f(x) = 1.9 \) if \( x = -0.1 \), so \(-0.1 \leq x \leq 0.1 \).
2.3.23. \( f(x) = 1.1 \) if \( x = \sqrt{1.1} = 1.049 \) and \( f(x) = 0.949 \) if \( x = -0.1 \), so \(-0.949 \leq x \leq 1.049 \).
2.3.25. Any value of \( x \geq 0 \) works.
2.3.27.a. 

![Graph of a function](image)

b. The slope on the central part is 10.

\[
S(x) = \begin{cases} 
-1 & \text{if } x \leq -0.1 \\
-1 + 10(x + 0.1) & \text{if } 0.1 < x < 0.1 \\
1 & \text{if } x \geq 0.1 
\end{cases}
\]

c. The input would have to be between -0.01 and 0.01.
2.3.29.

\[
\begin{align*}
4.8 & < M < 5.2 \\
4.8 & < 2.0V < 5.2 \\
2.4 & < V < 2.6.
\end{align*}
\]

\(V\) must be within 0.1 cm\(^3\) of 2.5 cm\(^3\) for \(M\) to be within 0.2 g of 5.0 g.

2.3.31.

\[
\begin{align*}
0.95 & < F < 1.05 \\
0.95 & < r^4 < 1.05 \\
0.987 & < r < 1.012.
\end{align*}
\]

The radius must be within about 1\% of 1.0 to guarantee a flow within 5\%.

2.3.33. Between 4.5\times 10^6 and 5.5\times 10^6. Tolerance is 5.0\times 10^7.

2.3.35. Between 8.789\times 10^5 and 10.742\times 10^5. Tolerance is 9.76\times 10^4.

2.3.37. Between 0.96 and 1.04 g/L. Tolerance is 0.04 g/L.

2.3.39. Between 491.52 and 532.48 g/L. Tolerance is 20.48 g/L.

2.3.41. Denote the function by \(f\). Then

\[
f(V) = \begin{cases} 
80 & \text{if } V < 50 \\
2V & \text{if } V \geq 50
\end{cases}
\]

2.3.43. We need that \(2V = V^*\) at \(V = V_0 = 50\). In this case, \(V^* = 100\).

2.3.45.

**a.**

The second seems more likely. The pitch is usually different in different directions.

**c.** Both sound awful.