1.11 Equilibria

MATHEMATICAL TECHNIQUES

♠ Solve the following equations. Check your answer by plugging in the value you found.
- **EXERCISE 1.11.1**
  \[ 2x + 3 = 7. \]
- **EXERCISE 1.11.2**
  \[ \frac{1}{2}z - 3 = 7. \]
- **EXERCISE 1.11.3**
  \[ 2x + 3 = 3x + 7. \]
- **EXERCISE 1.11.4**
  \[ -3y + 5 = 8 + 2y. \]
- **EXERCISE 1.11.5**
  \[ 2(5(x - 1) + 3) = 5(2(x - 2) + 7). \]
- **EXERCISE 1.11.6**
  \[ 2(4(x - 1) + 3) = 5(2(x - 2) + 7). \]

♠ Solve the following equations for the given variable, treating the others as constant parameters.
- **EXERCISE 1.11.7**
  Solve \[ 2x + b = 7 \] for \( x \).
- **EXERCISE 1.11.8**
  Solve \[ mx + 3 = 7 \] for \( x \).
- **EXERCISE 1.11.9**
  Solve \[ 2x + b = mx + 7 \] for \( x \). Are there any values of \( b \) or \( m \) for which this has no solution?
- **EXERCISE 1.11.10**
  Solve \[ mx + b = 3x + 7 \] for \( x \). Are there any values of \( b \) or \( m \) for which this has no solution?

♠ Find the equilibria of the following updating functions from their graphs. Label the coordinates of the equilibria.
- **EXERCISE 1.11.11**

![Graph 1](image1.png)

- **EXERCISE 1.11.12**

![Graph 2](image2.png)

- **EXERCISE 1.11.13**
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- **EXERCISE 1.11.14**

![Graph of a system](attachment:graph.png)

♠ Graph the following discrete-time dynamical systems. Solve for the equilibria algebraically, and identify equilibria and the regions where the updating function lies above the diagonal on your graph.
- **EXERCISE 1.11.15**
  \( c_{t+1} = 0.5c_t + 8.0, \text{ for } 0 \leq c_t \leq 30. \)
- **EXERCISE 1.11.16**
  \( b_{t+1} = 3b_t, \text{ for } 0 \leq b_t \leq 10. \)
- **EXERCISE 1.11.17**
  \( b_{t+1} = 0.3b_t, \text{ for } 0 \leq b_t \leq 10. \)
- **EXERCISE 1.11.18**
  \( b_{t+1} = 2.0b_t - 5.0, \text{ for } 0 \leq b_t \leq 10. \)

♠ Sketch graphs of the following updating functions over the given range and mark the equilibria. Find the equilibria algebraically if possible.
- **EXERCISE 1.11.19**
  \( f(x) = x^2, \text{ for } 0 \leq x \leq 2. \)
- **EXERCISE 1.11.20**
  \( g(y) = y^2 - 1, \text{ for } 0 \leq y \leq 2. \)
- **EXERCISE 1.11.21**
  \( h(z) = e^{-z}, \text{ for } 0 \leq z \leq 2. \) (This cannot be solved algebraically).
- **EXERCISE 1.11.22**
  \( F(x) = \ln(x) + 1, \text{ for } 0 \leq x \leq 2. \) (This cannot be solved algebraically, but you can guess the answer).

♠ Find the equilibria of the following discrete-time dynamical systems. Compare with the results of your cobweb diagram from the earlier problem.
- **EXERCISE 1.11.23**
  \( v_{t+1} = 1.5v_t, \text{ (as in exercise 1.10.11).} \)
- **EXERCISE 1.11.24**
  \( l_{t+1} = l_t - 1.7, \text{ (as in exercise 1.10.12).} \)
- **EXERCISE 1.11.25**
  \( x_{t+1} = 2x_t - 1, \text{ (as in exercise 1.10.15).} \)
- **EXERCISE 1.11.26**
  \( z_{t+1} = 0.9z_t + 1, \text{ (as in exercise 1.10.16).} \)
- **EXERCISE 1.11.27**
  \( w_{t+1} = -0.5w_t + 3, \text{ (as in exercise 1.10.17).} \)
• **EXERCISE 1.11.28**
  \[ x_{t+1} = 4 - x_t \] (as in exercise 1.10.18).

• **EXERCISE 1.11.29**
  \[ x_{t+1} = \frac{x_t}{x_t + x_0} \] (as in exercise 1.10.19).

• **EXERCISE 1.11.30**
  \[ x_{t+1} = \frac{x_t}{x_t - 1} \] (as in exercise 1.10.20).

**APPLICATIONS**

★ Find the lung updating function with the following parameter values, and compute the equilibrium. Check that it matches the formula \( c^* = \gamma \).

• **EXERCISE 1.11.31**
  \[ V = 2.0 \text{ L}, \; W = 0.5 \text{ L}, \; \gamma = 5.0 \text{ mmol}/\text{L}, \; c_0 = 1.0 \text{ mmol}/\text{L} \] (as in exercise 1.10.21).

• **EXERCISE 1.11.32**
  \[ V = 1.0 \text{ L}, \; W = 0.1 \text{ L}, \; \gamma = 8.0 \text{ mmol}/\text{L}, \; c_0 = 4.0 \text{ mmol}/\text{L} \] (as in exercise 1.10.22).

• **EXERCISE 1.11.33**
  \[ V = 1.0 \text{ L}, \; W = 0.9 \text{ L}, \; \gamma = 5.0 \text{ mmol}/\text{L}, \; c_0 = 9.0 \text{ mmol}/\text{L} \] (as in exercise 1.10.23).

• **EXERCISE 1.11.34**
  \[ V = 1.0 \text{ L}, \; W = 0.2 \text{ L}, \; \gamma = 1.0 \text{ mmol}/\text{L}, \; c_0 = 9.0 \text{ mmol}/\text{L} \] (as in exercise 1.10.24).

★ Find the equilibrium population of bacteria in the following cases with supplementation. Graph the updating function for each.

• **EXERCISE 1.11.35**
  A population of bacteria has per capita reproduction \( r = 0.6 \) and \( 1.0 \times 10^6 \) bacteria are added each generation (as in exercise 1.10.33).

• **EXERCISE 1.11.36**
  A population of bacteria has per capita reproduction \( r = 0.2 \) and \( 5.0 \times 10^6 \) bacteria are added each generation (as in exercise 1.10.34).

• **EXERCISE 1.11.37**
  A population of bacteria has per capita reproduction \( r = 0.5 \) and \( S \) bacteria are added each generation. What happens to the equilibrium when \( S \) is large. Does this make biological sense?

• **EXERCISE 1.11.38**
  A population of bacteria has per capita reproduction \( r = 0.5 \) and \( 1.0 \times 10^6 \) bacteria are added each generation. What happens to the equilibrium if \( r = 0 \)? What happens if \( r \) is close to 1? Do these results make biological sense?

★ Find the equilibrium concentration of chemical in the lung in the following models that include absorption.

• **EXERCISE 1.11.39**
  The situation described in exercise 1.10.31. How does the equilibrium compare with \( \gamma \)?

• **EXERCISE 1.11.40**
  The situation described in exercise 1.10.32. How does the equilibrium compare with \( \gamma \)?

★ Find the equilibrium concentration of salt in a lake in the following cases. Describe the result in words by comparing the equilibrium salt level with the salt level of the water flowing in.

• **EXERCISE 1.11.41**
  The situation described in exercise 1.10.35.

• **EXERCISE 1.11.42**
  The situation described in exercise 1.10.36.

• **EXERCISE 1.11.43**
  The situation described in exercise 1.10.37.

• **EXERCISE 1.11.44**
  The situation described in exercise 1.10.38.

★ A lab is growing and harvesting a culture of valuable bacteria described by the updating function

\[ b_{t+1} = rb_t - h. \]

The bacteria have per capita reproduction \( r \), and \( h \) are harvested each generation.
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• EXERCISE 1.11.45
  Suppose that \( r = 1.5 \) and \( h = 1.0 \times 10^6 \) bacteria. Sketch the updating function, and find the equilibrium both algebraically and graphically.

• EXERCISE 1.11.46
  Without setting \( r \) and \( h \) to particular values, find the equilibrium algebraically. Does the equilibrium get larger when \( h \) gets larger? Does it get larger when \( r \) gets larger? If the answers seem odd (as they should), look at a cobweb diagram to try to figure out why.
Chapter 4

Answers

1.11.1. $2x = 7 - 3 = 4$, so $x = 4/2 = 2$. Plugging in, $2 \cdot 2 + 3 = 7$.

1.11.3. $2x - 3x = 7 - 3 = 4$, so $-x = 4$ or $x = -4$. Plugging in, $2 \cdot (-4) + 3 = -5 = 3 \cdot (-4) + 7$.

1.11.5. Multiplying out, we get $10x + 1 = 10x - 3$. This has no solution.

1.11.7. $2x = 7 - b$, so $x = \frac{7 - b}{2}$.

1.11.9. $(2 - m)x = 7 - b$, so $x = \frac{7 - b}{2 - m}$. There is no solution if $m = 2$. However, if $m = 2$ and $b = 7$, both sides are identical and any value of $x$ works.

1.11.11. The equilibrium seems to be at about 1.3.

1.11.13. The equilibria seem to be at about 0.0 and 7.5.

1.11.15. The equilibrium is where $c^* = 0.5c^* + 8.0$ or $c^* = 16.0$.

1.11.17. The equilibrium is $b^* = 0$. 
1.11.19. \( f(x) = x \) when \( x^2 = x \), or \( x^2 - x = 0 \), or \( x(x - 1) = 0 \) which has solutions at \( x = 0 \) and \( x = 1 \).

1.11.23. \( v^* = 1.5v^* \) if \( v^* = 0 \).
1.11.25. \( x^* = 2x^* - 1 \) has solution \( x^* = 1 \).
1.11.27. \( w^* = -0.5w^* + 3 \) has solution \( w^* = 2 \).
1.11.29. \( x^* = \frac{-1}{1 + x^*} \) has solution \( x^* = 0 \).
1.11.31. The updating function is \( c_{t+1} = 0.75c_t + 1.25 \). Solving for the equilibrium, we find \( c^* = 0.75c^* + 1.25 \), or \( 0.25c^* = 1.25 \) or \( c^* = 5.0 \). This matches the value of \( \gamma \).
1.11.33. The updating function is \( c_{t+1} = 0.1c_t + 4.5 \). Solving for the equilibrium, we find \( c^* = 0.1c^* + 4.5 \), or \( 0.9c^* = 4.5 \) or \( c^* = 5.0 \). This matches the value of \( \gamma \).
1.11.35. The updating function is \( b_{t+1} = 0.6b_t + 1.0 \times 10^6 \). The equilibrium satisfies \( b^* = 0.6b^* + 1.0 \times 10^6 \) or \( 0.4b^* = 1.0 \times 10^6 \) or \( b^* = 2.5 \times 10^6 \).
1.11.37. The updating function is \( b_{t+1} = 0.5b_t + S \). The equilibrium satisfies \( b^* = 0.5b^* + S \), or \( 0.5b^* = S \) or \( b^* = 2S \). The equilibrium becomes larger when \( S \) is large. This makes sense because the population will be larger when more bacteria are added.
1.11.39. The updating function is \( c_{t+1} = 0.75c_t + 0.875 \). The equilibrium satisfies \( c^* = 0.75c^* + 0.875 \) or \( 0.25c^* = 0.875 \) or \( c^* = 1.75 \). This equilibrium concentration is much lower than the ambient concentration of \( \gamma = 5.0 \), due to the absorption.
1.11.41. The updating function is \( s_{t+1} = 0.99s_t + 0.0099 \). The equilibrium equation is \( s^* = 0.99s^* + 0.0099 \) or \( 0.01s^* = 0.0099 \) or \( s^* = 0.99 \). The water ends up almost exactly like the water that flows in.
1.11.43. The updating function is \( s_{t+1} = s_t + 0.01 \). The equilibrium equation is \( s^* = s^* + 0.01 \) which has no solution. This lake has no equilibrium and will get saltier and saltier.
1.11.45. The updating function is \( b_{t+1} = 1.5b_t - 1.0 \times 10^6 \), and the equilibrium is \( b^* = 2.0 \times 10^6 \).