Problem 1. Let $\alpha_n \to 0$, $x_n = \mathcal{O}(\alpha_n)$ and $y_n = \mathcal{O}(\alpha_n)$. Show that $x_n y_n = o(\alpha_n)$.

Problem 2.
(a) Write the Taylor expansion of $\ln(1+x)$ about $x = 0$ with the Lagrange form of the residual term.
(b) Assume the Taylor series for $\ln(1+x)$ is truncated after the term involving $x^{10}$ and is used to approximate the number $\ln 2$. What bound on the error can be given?

Problem 3. Consider the number $x = 2^6 + 2^{-16} + 2^{-19}$.
(a) Write $x$ in scientific base 2 (binary) notation of the form $1.b_1b_2b_3\ldots b_n \times 2^S$.
(b) If IEEE single precision is used, the number of bits above is limited to 23. What is $x_+$ (floating point number immediately above $x$) and $x_-$ (floating point number immediately below $x$).
(c) What is $\text{fl}(x)$ (the floating point representation of $x$, assuming round to nearest)
(d) What is machine precision $\epsilon$ in this system?
(e) Verify that the relative error between $x$ and $\text{fl}(x)$ is less than machine precision.

Problem 4. Halley’s method for solving $f(x) = 0$ uses the iteration formula

$$x_{n+1} = x_n - \frac{f_n f_n'}{(f_n')^2 - (f_n f_n'')/2},$$

where $f_n = f(x_n)$ and so on. Show that this formula results from applying Newton’s method to the function $f/\sqrt{f'}$.

Problem 5. Consider the polynomial $p(z) = z^4 + 2z^3 + 3z^2 + 3z + 2$.
(a) Compute $p(2)$ using Horner’s method
(b) Compute $p'(2)$ using Horner’s method
(c) Write $p(z)$ in the form $p(z) = (z - 2)q(z) + r$, specifying $q(z)$ and $r$.

Problem 6. Consider a smooth function $f$ with the following values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
</tbody>
</table>

(a) Write the polynomial interpolating $f(x)$ at the first three nodes in Lagrange form.
(b) Use divided differences to find the polynomial $p(x)$ interpolating $f(x)$ in Newton form.
(c) Assuming all derivatives of $f$ are available, give an expression of the interpolation error $f(t) - p(t)$ for some $t \in [-1, 2]$. 
Problem 7. Let $f(x)$ be a function of $x$ and $x_0, \ldots, x_n$ be $n+1$ distinct nodes. For $k = 0, \ldots, n$, let $p_k$ be the polynomial interpolating $f$ at the nodes $x_0, x_1, \ldots, x_k$. Let $q$ be the polynomial interpolating $f$ at the nodes $x_1, \ldots, x_n$. Show that:

$$p_n(x) = q(x) + \frac{x - x_n}{x_n - x_0}(q(x) - p_{n-1}(x)).$$