Example 5.1.2. *Le her*. The game of *le her* ("the gentleman" in 17th-century French) is a two-person game played with a standard 52-card deck, and we will continue to refer to the two players as player 1 and player 2. Cards are ranked from lowest to highest in the order A, 2, 3, . . . , 10, J, Q, K, and suits are ignored. A card is dealt face down to each player, and each player may look only at his own card. The object of the game is to have the higher-ranking card at the end of play. First, player 1, if he is not satisfied with his card, can require that player 2 exchange cards with him. The only exception to this rule occurs when player 2 has a king (K), in which case the exchange is void. Second, player 2, if he is not satisfied with his card, whether it be his original card or a new card obtained in exchange with player 1, can exchange it for the next card in the deck. The only exception to this rule occurs when the next card is a king, in which case the exchange is void. This completes the game, and the winner is the player with the higher-ranked card, with player 2 winning in the case of a tie. The game pays even money.

It will be convenient to define the *ranks* of the cards A, 2, 3, . . . , 10, J, Q, K as 1, 2, 3, . . . , 10, 11, 12, 13, respectively. Let us denote by $X$, $Y$, and $Z$ the ranks of the card dealt to player 1, the card dealt to player 2, and the next card in the deck, respectively. Player 1’s strategies correspond to the subsets $S \subseteq \{1, 2, \ldots, 13\}$. Given such an $S$, player 1 exchanges his card with that of player 2 if and only if $X \in S$. Player 2’s strategies correspond to the subsets $T \subseteq \{1, 2, \ldots, 13\}$. Given such a $T$, if player 1 fails to exchange his card with that of player 2, player 2 exchanges his card with the next card in the deck if and only if $Y \in T$. Of course, if player 1 exchanges his card with that of player 2, then player 2’s decision is clear: He keeps his new card if $X \geq Y$ and exchanges it for the next card in the deck otherwise. Thus, we have a $2^{13} \times 2^{13}$ matrix game.

It is intuitively clear that the only reasonable strategies are of the form $S_i := \{1, \ldots, i\}$ and $T_j := \{1, \ldots, j\}$ for $i, j = 0, 1, \ldots, 13$ (of course, $S_0 = T_0 := \emptyset$). It can be shown that every other strategy is strictly dominated by at least one of these (see Problem 5.9 on p. 195).

Let $B_{ij}$ denote the event that player 1 wins when player 1 uses strategy $S_i$ and player 2 uses strategy $T_j$ for $i, j = 0, 1, \ldots, 13$. We evaluate $P(B_{ij})$ by conditioning on $\{X = k, Y = l\}$. There are three cases to consider.

Case 1. $k \leq l$. Here player 1 exchanges his card with that of player 2, provided player 2 does not have a king. The only case in which player 1 can win is $k < l < 13$, which forces player 2 to exchange his new card with the next card in the deck. Player 1 wins if $Z < l$ or if $Z = 13$. Therefore,

$$P(B_{ij} \mid X = k, Y = l) = P(Z < l \text{ or } Z = 13 \mid X = k, Y = l) 1_{\{k < l < 13\}}$$

$$= \left( \frac{4(l - 1) - 1}{50} + \frac{4}{50} \right) 1_{\{k < l < 13\}}$$

$$= \frac{4l - 1}{50} 1_{\{k < l < 13\}}. \quad (5.7)$$
Case 2. $k > i$, $l \leq j$. Here player 1 keeps his card, while player 2 exchanges his card with the next card in the deck. Player 1 wins if $Z < k$ or if $Z = 13$ and $k > l$. Therefore,
\[
P(B_{ij} \mid X = k, Y = l) = P(Z < k \mid X = k, Y = l) + P(Z = 13 \mid X = k, Y = l) 1_{\{k > l\}}
\]
\[= \frac{4(k - 1) - 1_{\{k > l\}}}{50} + \frac{4 - \delta_{k,13}}{50} 1_{\{k > l\}}
\]
\[= \frac{4(k - 1) + (3 - \delta_{k,13}) 1_{\{k > l\}}}{50}.
\] (5.8)

Case 3. $k > i$, $l > j$. Here both players keep their cards, so
\[
P(B_{ij} \mid X = k, Y = l) = 1_{\{k > l\}}.
\]

It follows that
\[
P(B_{ij}) = \sum_{k=1}^{i} \sum_{l=1}^{13} \frac{4}{52} \frac{4l - 1}{51} 1_{\{k < l < 13\}}
\]
\[+ \sum_{k=i+1}^{13} \sum_{l=1}^{j} \frac{4}{52} \frac{4 - \delta_{k,l}}{51} \frac{4(k - 1) + (3 - \delta_{k,13}) 1_{\{k > l\}}}{50}
\]
\[+ \sum_{k=i+1}^{13} \sum_{l=j+1}^{13} \frac{4}{52} \frac{4}{51} 1_{\{k > l\}}.
\] (5.9)

Of course, this formula could be further simplified. For example, the first double sum could be written as a cubic polynomial in $i$. However, there is no need to do this, since our only concern is with the numerical evaluation of (5.9), and this is most reliably done by computer. The payoff matrix $A$ for this game has $(i, j)$ entry
\[
a_{ij} = 2P(B_{ij}) - 1.
\] (5.10)

The full matrix, multiplied by $(52)_3/2^3 = 16.575$, is displayed in Table 5.1.

The payoff matrix can be reduced considerably using strict dominance. Examining it, we see that strategies 0–4 for player 1 are strictly dominated by strategy 5 for player 1, and that strategies 8–13 for player 1 are strictly dominated by strategy 7 for player 1. Eliminating the strictly dominated rows, we are left with the $3 \times 14$ payoff matrix corresponding to the shaded rows in Table 5.1.

Next, within the shaded rows in Table 5.1 strategies 0–6 for player 2 are strictly dominated by strategy 7 for player 2, and strategies 9–13 for player 2 are strictly dominated by strategy 8 (or 7) for player 2. Eliminating the strictly dominated columns, we are left with
Finally, in (5.11) strategy 5 for player 1 is strictly dominated by strategy 6 for player 1, so we end up with the $2 \times 2$ payoff matrix, multiplied by 16,575, of

\[
\begin{pmatrix}
7 & 8 \\
5 & \begin{pmatrix} 105 & 221 \\ 393 & 429 \end{pmatrix} \\
6 & \begin{pmatrix} 105 & 221 \\ 393 & 429 \end{pmatrix} \\
7 & \begin{pmatrix} 105 & 221 \\ 393 & 429 \end{pmatrix}
\end{pmatrix}.
\]

In Example 5.2.6 on p. 183 we will find the optimal strategies for players 1 and 2. ♠
Table 5.1 Payoff matrix, multiplied by 16,575, for the game of le her. All rows except the shaded ones are strictly dominated.

\[
\begin{array}{cccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
2 & 1,173 & 333 & -507 & -1,167 & -1,643 & -1,935 & -2,043 & -1,967 & -1,707 & -1,263 & -635 & 177 & 1,173 & 2,357 \\
3 & 1,889 & 1,205 & 521 & -163 & -671 & -995 & -1,135 & -1,091 & -863 & -451 & 145 & 925 & 1,889 & 3,041 \\
5 & 2,525 & 2,105 & 1,685 & 1,265 & 845 & 425 & 173 & 105 & 221 & 521 & 1,005 & 1,673 & 2,525 & 3,565 \\
6 & 2,413 & 2,101 & 1,789 & 1,477 & 1,165 & 853 & 541 & 393 & 429 & 649 & 1,053 & 1,641 & 2,413 & 3,373 \\
7 & 1,993 & 1,733 & 1,553 & 1,333 & 1,113 & 893 & 673 & 453 & 393 & 517 & 825 & 1,317 & 1,993 & 2,857 \\
8 & 1,249 & 1,105 & 961 & 817 & 673 & 529 & 385 & 241 & 97 & 109 & 305 & 685 & 1,249 & 2,001 \\
\end{array}
\]