Math 3210-3
HW 22
Solutions

Properties of the Riemann Integral

1. A function \( f \) on \([a, b]\) is called a step function if there exists a partition \( P = \{a = u_0 < u_1 < \cdots < u_m = b\} \) of \([a, b]\) such that \( f \) is constant on each interval \((u_{j-1}, u_j)\), say \( f(x) = c_j \) for \( x \in (u_{j-1}, u_j) \).

   (a) Show that a step function \( f \) is integrable and evaluate \( \int_a^b f \).
   
   \textbf{Proof:} Let \( P = \{x_0, x_1, \ldots, x_n\} \) be a partition of \([a, b]\) such that \( f \) is constant on \((x_{i-1}, x_i)\) for each \( i = 1, \ldots, n \). Then we can say \( f(x) = c_i \) for \( x \in (x_{i-1}, x_i) \).
   
   I claim that \( f \) is integrable on \([x_{i-1}, x_i]\) for each \( i = 1, \ldots, n \). To prove this claim, notice that if \( g(x) = c_i \) for all \( x \in [x_{i-1}, x_i] \), then \( f(x) = g(x) \) except possibly at two points. Also \( g \) is monotonic, so by Theorem 98, \( g \) is integrable on \([x_{i-1}, x_i]\). By problem 6 on homework 21, we can conclude that \( f \) is integrable on \([x_{i-1}, x_i]\) for each \( i = 1, \ldots, n \). Thus by Theorem 100 \( f \) is integrable on \([a, b]\) and \( \int_a^b f = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f = \sum_{i=1}^{n} c_i \Delta x_i \).

2. Prove that if \( f \) is integrable on \([a, b]\) then so is \( f^2 \). (Hint: If \( |f(x)| \leq M \) for all \( x \in [a, b] \), then show \( |f^2(x) - f^2(y)| \leq 2M|f(x) - f(y)| \) for all \( x, y \in [a, b] \). Then use this to estimate \( U(f^2, P) - L(f^2, P) \) in terms of \( U(f, P) - L(f, P) \) for a given partition \( P \).)

   \textbf{Proof:} Let \( \epsilon > 0 \). Since \( f \) is bounded. Let \( M \in \mathbb{R} \) such that \( |f(x)| < M \) for all \( x \in [a, b] \). From Theorem 97, there is a partition \( P \) of \([a, b]\) such that \( U(f, P) - L(f, P) < \frac{\epsilon}{2M} \). I claim that with this partition we have \( U(f^2, P) - L(f^2, P) < \epsilon \), so by Theorem 97, \( f^2 \) will be integrable. To this end, notice that \( |f^2(x) - f^2(y)| = |f(x) + f(y)||f(x) - f(y)| \leq (|f(x)| + |f(y)|)|f(x) - f(y)| < 2M|f(x) - f(y)| \) for all \( x, y \in [a, b] \).

   Thus we have
   \[
   U(f^2, P) - L(f^2, P) = \sum |f^2(x_i) - f^2(y_i)||x_i - y_i| \\
   \leq \sum 2M|f(x_i) - f(y_i)||x_i - y_i| \\
   = 2M(U(f, P) - L(f, P)) \\
   < 2M\frac{\epsilon}{2M} = \epsilon
   \]

   Therefore \( f^2 \) is integrable.

3. Prove that if \( f \) and \( g \) are integrable on \([a, b]\), then so is \( fg \). (Hint: Use the previous problem to write \( fg \) as the sum of two functions which you know are integrable.)

   \textbf{Proof:} Notice that \( f \) and \( g \) are integrable, so \( f + g, f^2, g^2 \), and \((f + g)^2\) are integrable by the previous problem and Theorem 100. Thus \( \frac{1}{2}[(f + g)^2 - f^2 - g^2] = fg \) is integrable.
4. Find an example of a function \( f : [0, 1] \to \mathbb{R} \) such that \( f \) is not integrable on \([0, 1]\) by \(|f|\) is integrable on \([0, 1]\).

**Proof:** Let \( f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases} \). Let \( P = \{x_0, \ldots, x_n\} \) be any partition of \([0, 1]\). Then \( M_i = 1 \) and \( m_i = -1 \) for all \( i = 1, \ldots, n \). Thus \( U(f, P) = 1 \) and \( L(f, P) = -1 \) for all \( P \). Thus \( f \) is not integrable.

On the other hand, \(|f|(x) = 1\) for all \( x \in [0, 1] \). Since \(|f|\) is a continuous function, \(|f|\) is integrable on \([0, 1]\) by Theorem 98.

5. Suppose that \( f \) and \( g \) are continuous function on \([a, b]\) such that \( \int_a^b f = \int_a^b g \). Prove that there exists \( x \in [a, b] \) such that \( f(x) = g(x) \).

**Proof:** Let \( h(x) = f(x) - g(x) \). Then \( h \) is continuous by Theorem 71, and \( h \) assumes its max and min values on \([a, b]\) by Corollary 5. In other words, there is some \( x_1, x_2 \in [a, b] \) such that \( h(x_1) = m \leq h(x) \leq M = h(x_2) \) for all \( x \in [a, b] \). If \( P \) is the partition \( P = \{a, b\} \), then \( L(h, P) = m(b-a) \) and \( U(h, P) = M(b-a) \), so we have \( L(h, P) = m(b-a) \leq \int_a^b h \leq M(b-a) = U(f, P) \Rightarrow h(x_1) = m \leq \frac{1}{b-a} \int_a^b h < M = h(x_2) \). Thus we can apply the intermediate value theorem to get some \( x \in [a, b] \) such that \( h(x) = \frac{1}{b-a} \int_a^b h = \frac{1}{b-a} \int_a^b (f - g) = 0 \). Thus \( f(x) = g(x) \) for some \( x \in [a, b] \).