L’Hospital’s Rule

1. Evaluate the following limits.

(a) \[ \lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x} = \lim_{x \to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6} = \frac{1}{3} \]

(b) \[ \lim_{x \to 0} \frac{\sin x - x}{e^x - 1} = \lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \]

(c) \[ \lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0 \]

(d) \[ \lim_{x \to 0^+} \frac{\log \sin x}{\log x} = \lim_{x \to 0^+} \frac{\frac{1}{\sin x} \cos x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{x \cos x}{\sin x} = \lim_{x \to 0^+} \frac{\cos x - x \sin x}{\cos x} = 1 \]

(e) \[ \lim_{x \to \infty} \frac{(\log x)^2}{x} = \lim_{x \to \infty} 2 \log x \left(\frac{1}{x} \right) = \lim_{x \to \infty} \frac{2 \log x}{x} = \lim_{x \to \infty} \frac{2}{x} = 0 \]

(f) \[ \lim_{x \to 0^+} x^{2x} \]

Let \( y = x^{2x} \). Then \( \log y = 2x \log x \), so let’s compute \( \lim_{x \to 0^+} 2x \log x \). We have \( \lim_{x \to 0^+} 2x \log x = \lim_{x \to 0^+} \log x \lim_{x \to 0^+} \frac{2x}{x} = \lim_{x \to 0^+} \frac{2x}{x} = \lim_{x \to 0^+} \frac{8x^2}{4x^2} = \lim_{x \to 0^+} \frac{8x^2}{4x^2} = 0 \). So \( \lim_{x \to 0^+} x^{2x} = \lim_{x \to 0^+} e^{2x \log x} = e^{\lim_{x \to 0^+} 2x \log x} = e^0 = 1 \).

(g) \[ \lim_{x \to 0} x \ln x \]

\[ \lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} \text{ and now we can use L’Hospital’s rule. We have } \lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0} \frac{-x^2}{x} = \lim_{x \to 0} -x = 0. \]

(h) \[ \lim_{x \to \infty} (\sqrt{x + 1} - \sqrt{x}) \]

\[ \lim_{x \to \infty} (\sqrt{x + 1} - \sqrt{x}) = \lim_{x \to \infty} \frac{(\sqrt{x + 1} - \sqrt{x})(\sqrt{x + 1} + \sqrt{x})}{\sqrt{x + 1} + \sqrt{x}} = \lim_{x \to \infty} \frac{x + 1 - x}{\sqrt{x + 1} + \sqrt{x}} = \lim_{x \to \infty} \frac{1}{\sqrt{x + 1} + \sqrt{x}} = 0. \]

2. Indicate what is wrong with the following result:

\[ \lim_{x \to 1} \frac{2x^2 - x - 1}{3x^2 - 5x + 2} = \lim_{x \to 1} \frac{4x - 1}{6x - 5} = \lim_{x \to 1} \frac{4}{6} = \frac{2}{3} \]

The problem is that \( \lim_{x \to 1} \frac{4x - 1}{6x - 5} = \frac{3}{1} = 3 \) which is not an indeterminate form, so we should not have applied L’Hospital’s rule another time to get the last two equalities.
3. Prove that if \( r > 0 \) and \( x > 1 \), then \( \ln x \leq \frac{x^r - 1}{r} \). Hint: Use Cauchy’s form of the Mean Value Theorem with \( f(x) = \ln x \) and \( g(x) = x^r \).

**Proof:** Let \( f(y) = \ln y \) and \( g(y) = y^r \) for \( y \in [1, x] \) for some \( x > 1 \). Then \( f \) and \( g \) are continuous and differentiable on \([1, x]\), so by Cauchy’s form of the Mean Value Theorem there is some \( c \in (1, x) \) such that

\[
\frac{f(x) - f(1)}{g(x) - g(1)} = \frac{f'(c)}{g'(c)}.
\]

Thus

\[
\frac{\ln x}{x^r - 1} = \frac{\frac{1}{c}}{ec^{cr}} = \frac{1}{rc} \leq \frac{1}{r} \text{ since } c \in (1, x).
\]

Therefore \( \ln x \leq \frac{x^r - 1}{r} \).

4. Prove that \( 1 + x^2 \leq e^{x^2} \) for all \( x \in \mathbb{R} \).

**Proof:** First, if \( x = 0 \), clearly \( 1 + 0 \leq e^0 = 1 \), so we may assume \( x \neq 0 \). Let \( f(y) = 1 + y^2 \) and \( g(y) = e^{y^2} \) for \( y \in [0, x] \). (Notice that if \( x < 0 \), we could write our interval as \([x, 0]\).) We know \( f \) and \( g \) are continuous and differentiable on \([0, x]\), so by Cauchy’s Mean Value Theorem, there is some \( c \) between 0 and \( x \) such that

\[
\frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(c)}{g'(c)}.
\]

Thus

\[
\frac{1 + x^2 - 1}{e^{x^2} - 1} = \frac{2c}{e^{c^2}(2c)} = \frac{1}{e^{c^2}} \leq \frac{1}{e^0} = 1.
\]

Notice this is true for any \( x \neq 0 \). Therefore \( x^2 \leq e^{x^2} - 1 \implies 1 + x^2 \leq e^{x^2} \) for all \( x \in \mathbb{R} \).