Subsequences

1. For each sequence, find the set $S$ of subsequential limits, the lim sup, and the lim inf. No proofs are needed.

   (a) $w_n = (-1)^n$: $S = \{0\}$, $\limsup w_n = \liminf w_n = 0$
   (b) $(x_n) = (0, 1, 2, 0, 1, 3, 0, 1, 4, \ldots)$: $S = \{0, 1\}$, $\limsup x_n = +\infty$, $\liminf x_n = 0$
   (c) $y_n = n[2 + (-1)^n]$: $S = \emptyset$, $\limsup y_n = +\infty$, $\liminf y_n = +\infty$
   (d) $z_n = (-n)^n$: $S = \emptyset$, $\limsup z_n = +\infty$, $\liminf z_n = -\infty$
   (e) $b_n = \sin \left( \frac{n\pi}{3} \right)$: $S = \left\{ -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2} \right\}$, $\limsup b_n = \frac{\sqrt{3}}{2}$, $\liminf b_n = -\frac{\sqrt{3}}{2}$

2. If $\limsup a_n$ and $\limsup b_n$ are finite, prove that $\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n$.

   Proof: Let $c_n = a_n + b_n$ and let $a = \limsup a_n$, $b = \limsup b_n$ and $c = \limsup c_n$. Let $\epsilon > 0$. Then by Theorem 64, there exists $N_1$ such that for all $n > N_1$, $a_n \leq a + \frac{\epsilon}{2}$. Similarly, there exists $N_2$ such that for all $n > N_2$, $b_n \leq b + \frac{\epsilon}{2}$. Thus we have $c_n = a_n + b_n \leq a + b + \epsilon$ for all $\epsilon > 0$ and all $n > \max\{N_1, N_2\}$. Hence by Theorem 26, $c_n \leq a + b$ for all $n > \max\{N_1, N_2\}$. Let $(c_{n_k})$ be any subsequence in $(c_n)$ which converges to $c$. For $n_k > N$, $c_{n_k} \leq a + b$, so by Theorem 54, $c = \lim c_{n_k} \leq a + b$. Therefore $\limsup c_n \leq \limsup a_n + \limsup b_n$.

3. Let $(r_n)$ be an enumeration of the set $\mathbb{Q}$. Show that there exists a subsequence $(r_{n_k})$ such that $\lim_{k \to \infty} r_{n_k} = +\infty$.

   Proof: Let $A_1 = \{n \in \mathbb{N} : r_n > 0\}$, and let $n_1 = \min A_1$. Let $A_2 = \{n \in \mathbb{N} : n > n_1$ and $r_n > r_{n_1} + 1\}$, and let $n_2 = \min A_2$. Continue in this way to get $n_3, n_4, \ldots$. So in general $n_k = \min\{n \in \mathbb{N} : k > n_{k-1}$ and $r_{n_k} > r_{n_{k-1}} + 1\}$. Then $(r_{n_k})$ is an increasing sequence. Also for $\epsilon = \frac{1}{k}$, $|r_{n_p} - r_{n_m}| > \epsilon$ for all $p, m$, which implies that $(r_{n_k})$ is not Cauchy, so it diverges and $r_{n_k} > 0$ for all $n_k$. Thus $\lim_{k \to \infty} r_{n_k} = +\infty$.