Note: Problems 4 and 7 are extra credit.

**Limit Theorems**

1. Suppose that \( \lim a_n = a \) and \( \lim b_n = b \). Let \( s_n = \frac{a_n^3 + 4a_n}{b_n^2 + 1} \). Prove that \( \lim s_n = \frac{a^3 + 4a}{b^2 + 1} \) carefully, using the limit theorems.

   **Proof:** We have the following:

   \[
   \lim s_n = \lim \frac{a_n^3 + 4a_n}{b_n^2 + 1} \quad \text{by definition}
   \]

   \[
   = \frac{\lim(a_n^3 + 4a_n)}{\lim(b_n^2 + 1)} \quad \text{by Theorem 50(4)}
   \]

   \[
   = \frac{\lim(a_n^3) + \lim(4a_n)}{\lim(b_n^2) + \lim 1} \quad \text{by Theorem 50(1)}
   \]

   \[
   = \frac{(\lim a_n)^3 + \lim(4a_n)}{(\lim b_n)^2 + \lim 1} \quad \text{by Theorem 50(3)}
   \]

   \[
   = \frac{(\lim a_n)^3 + 4 \lim a_n}{(\lim b_n)^2 + 1} \quad \text{by Theorem 50(2)}
   \]

   \[
   = \frac{a^3 + 4a}{b^2 + 1} \quad \text{by hypothesis}
   \]

\[\Box\]

2. (a) Verify \( 1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a} \) for \( a \neq 1 \).

   **Proof:** We will prove this by induction. Let \( n = 1 \). Then \( 1 + a = \frac{1-a^2}{1-a} = \frac{(1+a)(1-a)}{1-a} = 1 + a \).

   Hence \( n = 1 \) is true. Assume \( 1 + a + a^2 + \cdots + a^k = \frac{1-a^{k+1}}{1-a} \) for some \( k \in \mathbb{N} \). Then we have the following:

   \[
   1 + a + a^2 + \cdots + a^k + a^{k+1} = \frac{1-a^{k+1}}{1-a} + a^{k+1}
   \]

   \[
   = \frac{1-a^{k+1} + a^{k+1} - a^{k+2}}{1-a}
   \]

   \[
   = \frac{1-a^{k+2}}{1-a}
   \]

   Therefore \( 1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a} \) for \( a \neq 1 \).

\[\Box\]

(b) \( \clubsuit \) Find \( \lim_{n \to \infty}(1 + a + a^2 + \cdots + a^n) \) for \( |a| < 1 \).

   **Proof:** By Theorem 50, we see that \( \lim_{n \to \infty}(1 + a + a^2 + \cdots + a^n) = \lim_{n \to \infty} \frac{1-a^{n+1}}{1-a} = \frac{\lim_{n \to \infty}(1-a^{n+1})}{\lim_{n \to \infty}(1-a)} = \frac{1-\lim a^{n+1}}{1-\lim a}. \)

   Since \( |a| < 1 \), we know from problem 3(f), \( \lim a^n = 0 \), so \( \lim_{n \to \infty}(1 + a + a^2 + \cdots + a^n) = \frac{1}{1-a} \).

\[\Box\]
(c) Calculate \( \lim_{n \to \infty} (1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n}) \).

By part (b), \( \lim_{n \to \infty} \left(1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n}\right) = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \).

(d) What is \( \lim_{n \to \infty} (1 + a + a^2 + \cdots + a^n) \) for \( a \geq 1 \)?

If \( a \geq 1 \), \( \lim_{n \to \infty} (1 + a + a^2 + \cdots + a^n) = +\infty \). To prove this, recall that if \( a \geq 1 \), then \( a^n > a \) for all \( n \in \mathbb{N} \). So \( 1 + a + a^2 + \cdots + a^n > 1 + a + a + \cdots + a = 1 + na > n \). Therefore for any \( M \in \mathbb{R} \), by the Archimedean property, there is some \( n \in \mathbb{N} \) such that \( n > M \), but we also have that \( n < 1 + na < 1 + a + a + \cdots + a < 1 + a + a^2 + \cdots + a^n \). Therefore, \( \lim_{n \to \infty} (1 + a + a^2 + \cdots + a^n) = +\infty \).

### Monotone and Cauchy Sequences

3. Which of the following sequences are nondecreasing? nonincreasing? bounded? No proofs required.

(a) \( \frac{1}{n} \) Nonincreasing, Bounded

(b) \( \frac{(-1)^n}{n^2} \) Bounded

(c) \( n^5 \) Nondecreasing

(d) \( \frac{n}{\pi} \) Bounded

(e) \( (2)^n \) Neither

(f) \( \frac{n}{3^n} \) Nonincreasing, Bounded

4. Let \( (s_n) \) be a nondecreasing sequence of positive numbers and define \( \sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n} \). Prove that \( (\sigma_n) \) is a nondecreasing sequence.

**Proof:** We will prove this by induction. We know that \( 0 < s_1 \leq s_2 \), so \( s_1 + s_1 \leq s_1 + s_2 \), which implies that \( 2s_1 \leq s_1 + s_2 \). Thus \( s_1 \leq \frac{s_1 + s_2}{2} \). So \( n = 1 \) is true. Assume \( \sigma_k \) is true, that is \( \frac{s_1 + \cdots + s_{k+1}}{k+1} \leq \frac{s_1 + \cdots + s_{k+1}}{k+1} \). By hypothesis, we know \( s_n \geq s_j \) for all \( j < n \), so \( s_1 + s_2 + \cdots + s_{k+1} \leq s_{k+1} + \cdots + s_{k+1} \) where there are \( k+1 \) summands. So \( s_1 + s_2 + \cdots + s_{k+1} \leq s_{k+1}(k+1) \). Thus \( 0 \leq s_{k+2}(k+1) - s_1 - s_2 - \cdots - s_{k+1} \). Hence we have the following:

\[
\frac{s_1 + \cdots + s_{k+1}}{k+1} \leq \frac{s_1 + \cdots + s_{k+1}}{k+1} + \frac{s_{k+2}(k+1) - s_1 - s_2 - \cdots - s_{k+1}}{k+1} = \frac{(k+2)(s_1 + \cdots + s_{k+1}) + s_{k+2}(k+1) - s_1 - s_2 - \cdots - s_{k+1}}{(k+2)(k+1)} = \frac{s_1k + \cdots + s_{k+1}k + 2s_1 + \cdots + 2s_{k+1} + s_{k+2}(k+1) - s_1 - s_2 - \cdots - s_{k+1}}{(k+2)(k+1)} = \frac{s_1 + s_2k + \cdots + s_{k+1}k + s_1 + s_2 + \cdots + s_{k+1} + s_{k+2}(k+1)}{(k+2)(k+1)} = \frac{s_1(k+1) + s_2(k+1) + \cdots + s_{k+1}(k+1) + s_{k+2}(k+1)}{(k+1)(k+2)} = \frac{s_1 + s_2 + \cdots + s_{k+2}}{k+2} = \sigma_{k+2}
\]

Therefore \( (\sigma_n) \) is a nondecreasing sequence.

5. (a) Let \( s_1 = 1 \) and \( s_{n+1} = \frac{s_n + 1}{3} \) for \( n \geq 1 \).

   (a) Find \( s_2, s_3 \) and \( s_4 \).

   \( s_2 = \frac{4}{3}, s_3 = \frac{5}{9}, \) and \( s_4 = \frac{14}{27} \).
(b) Use induction to show that \( s_n > \frac{1}{2} \) for all \( n \in \mathbb{N} \).

**Proof:** We can see from part (a) that \( s_1 \) and \( s_2 \) are both \( > \frac{1}{2} \). Now assume \( s_k > \frac{1}{2} \) for some \( k \in \mathbb{N} \). Then we have \( s_k = \frac{s_{k-1}+1}{3} > \frac{1}{2} \), so

\[
\frac{s_{k-1}+1}{3} > \frac{1}{2} + 1
\]

\[
= \frac{1}{3}
\]

Thus \( s_n > \frac{1}{2} \) for all \( n \in \mathbb{N} \). \( \square \)

(c) Show that \( (s_n) \) is a nonincreasing sequence.

**Proof:** We need to show that \( s_n \geq s_{n+1} \) for all \( n \in \mathbb{N} \). From above, we see that \( s_1 = 1 > \frac{2}{3} = s_2 \), so \( n = 1 \) is true. Assume \( s_k > s_{k+1} \) for some \( k \in \mathbb{N} \). Then \( s_{k+1} = \frac{s_k+1}{3} \geq \frac{s_{k+1}+1}{3} = s_{k+2} \). Therefore \( s_n \) is a nonincreasing sequence. \( \square \)

(d) Show that \( \lim s_n \) exists and find \( \lim s_n \).

**Proof:** Since \( 0 < s_n < 1 \) for all \( n \in \mathbb{N} \), and since \( (s_n) \) is monotonic, by Theorem 59, \( (s_n) \) converges to a real number, so the limit exists. We know \( \lim s_n = \lim s_{n+1} \), so if \( s = \lim s_n \), we have

\[
s = \lim s_n
\]

\[
= \lim \frac{s_n + 1}{3}
\]

\[
= \frac{s + 1}{3}
\]

Thus \( s = \frac{s + 1}{3} \). We can solve for \( s \) to get \( s = \frac{1}{2} \).

\( \square \)

6. Let \( s_n = a_1 + a_2 + \cdots + a_n \) where each \( a_i \in \mathbb{R} \), and let \( t_n = |a_1| + |a_2| + \cdots + |a_n| \). Prove that if \( (t_n) \) is a bounded sequence then \( (s_n) \) converges.

**Proof:** Notice that we have:

\[
t_1 = |a_1|
\]

\[
t_2 = |a_1| + |a_2|
\]

\[
t_3 = |a_1| + |a_2| + |a_3|
\]

\[\vdots\]

\[
t_n = |a_1| + |a_2| + |a_3| + \cdots + |a_n|
\]

\[\vdots\]

So \( (t_n) \) is a nondecreasing sequence. By assumption it is bounded, so by the Monotone Convergence Theorem \( (t_n) \) converges. Thus \( (t_n) \) is Cauchy. So by definition given \( \epsilon > 0 \) there exists \( N \) such that for all \( n \geq m > N \), \( |t_n - t_m| < \epsilon \). But

\[
|t_n - t_m| = ||a_1| + |a_2| + \cdots + |a_n| - (|a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}| + \cdots + |a_m||
\]

\[
= ||a_{n+1}| + |a_{n+2}| + \cdots + |a_m||
\]

\[
< \epsilon
\]
Thus for this same $N$ value, if $n \geq m > N$ we have

$$|s_n - s_m| = |a_1 + \cdots + a_n - (a_1 + \cdots + a_m)|$$

$$= |a_{n+1} + a_{n+2} + \cdots + a_m|$$

$$\leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m|$$

$$= ||a_{n+1}| + |a_{n+2}| + \cdots + |a_m||$$

$$< \epsilon$$

Therefore $|s_n - s_m| < \epsilon$, so $(s_n)$ is Cauchy.

\[\square\]

7. If $|a_{n+1} - a_n| < 3^{-n}$ for all $n \in \mathbb{N}$, prove that $(a_n)$ is Cauchy, and then conclude that $(a_n)$ converges. (At some point in your proof problem 2(c) could be helpful.)

**Proof:** Let $\epsilon > 0$ and let $N = \frac{-\ln\left(\frac{2\epsilon}{3}\right)}{\ln 3}$. Then for all $n \geq m > N$ we have

$$|a_n - a_m| = |a_n - a_{n-1} + a_{n-1} - a_{n-2} + a_{n-2} - \cdots + a_{m+1} - a_m|$$

$$\leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{m+1} - a_m|$$

$$< 3^{-(n-1)} + 3^{-(n-2)} + \cdots + 3^{-m}$$

Since $n \geq m$ let $k \in \mathbb{N}$ with $n = m + k$. Then

$$|a_n - a_m| < 3^{-m}(3^{-(k-1)} + 3^{-(k-2)} + \cdots + 3^{-1} + 1)$$

$$< 3^{-m} \left[\sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i\right]$$

$$= 3^{-m} \cdot \frac{3}{2}$$

$$= \left(\frac{3}{2}\right) 3^{-m}$$

But $m > N$, so $|a_n - a_m| < \left(\frac{3}{2}\right) 3^{-N} = \epsilon$. Therefore $(a_n)$ is Cauchy, and since every Cauchy sequence in the reals converges, $(a_n)$ converges.

\[\square\]

Note: $\left(\frac{3}{2}\right) 3^{-N} = \epsilon \iff 3^{-N} = \frac{2\epsilon}{3} \iff -N \ln 3 = \ln \frac{2\epsilon}{3} \iff N = \frac{-\ln \frac{2\epsilon}{3}}{\ln 3}$.

8. Find an example of a sequence of real numbers satisfying each set of properties.

(a) Cauchy, but not monotone: Let $a_n = \frac{(-1)^n}{n}$

(b) Monotone, but not Cauchy: Let $a_n = n$

(c) Bounded, but not Cauchy: Let $a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$