Name: ________________________________

You may use your dictionary of definitions. Please use a pencil and keep your proofs neat and organized. Make sure you use complete “sentences,” and remember you need to give justifications for each step in your proofs. Each problem is worth one point. Write on this front page which problems you want graded. Notice that you have some choice as to which problems you will do. You may refer to theorems we proved in class by either stating the name of the theorem or giving a brief synopsis of the statement of the theorem.

Grade Problems: ________________________________
2. State the Fundamental Theorem of Calculus. (Either of the two.)

The First Fundamental Theorem of Calculus: Let \( f \) be integrable on \([a, b]\). For each \( x \in [a, b] \) let \( F(x) = \int_a^x f(t) \, dt \). Then \( F \) is uniformly continuous on \([a, b]\). Furthermore, if \( f \) is continuous at \( c \in [a, b] \) then \( F \) is differentiable at \( c \) and \( F'(c) = f(c) \).

The Second Fundamental Theorem of Calculus: If \( f \) is differentiable on \([a, b]\) and \( f' \) is integrable on \([a, b]\), then \( \int_a^b f = f(b) - f(a) \).

Do one of the following:

3. Prove that the polynomial function \( f_m(x) = x^3 - 3x + m \) never has two roots in \([0, 1]\), no matter what \( m \) may be.

Proof: Suppose \( f_m(x) \) has two roots in \([0, 1]\). Thus there is some \( x, y \in [0, 1] \) such that \( f_m(x) = f_m(y) = 0 \). By the mean value theorem, there is some \( c \in (x, y) \) such that \( f_m'(c) = \frac{f_m(y) - f_m(x)}{y - x} = 0 \). But \( f_m'(c) = 3c^2 - 3 = 3(c + 1)(c - 1) = 0 \) which implies that \( c = 1 \) or \( c = -1 \), a contradiction. Therefore \( f_m(x) \) never has two roots in \([0, 1]\). □

4. Let \( f(x) = \sin^{-1} x \). What is \( f'(x) \)? (Hint: At some point it might be helpful to complete the diagram below. And just writing the formula you might remember from Calculus does not count.)

Proof: Notice that \( f(x) \) is the inverse function of \( g(\theta) = \sin \theta \). So by the Inverse Function Theorem,

\[
 f'(x) = \frac{1}{g'(\theta)} = \frac{1}{\cos \theta} = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1 - x^2}}.
\]

\[\begin{array}{c}
\sqrt{1-x^2} \\
\hline
\theta \\
1 \\
x
\end{array}\]

Prove one of the following:

5. (a) If \( h(x) \geq 0 \) for all \( x \in [a, b] \), then \( \int_a^b h(x) \, dx \geq 0 \).

Proof: Let \( P \) be the partition \( P = \{a, b\} \). Then there exists \( x_1 \in [a, b] \) such that \( m = h(x_1) \leq h(x) \) for all \( x \in [a, b] \). Now \( m \geq 0 \), so \( 0 \leq L(h, P) = m(b - a) \leq \int_a^b h(x) \, dx \).

(b) If \( f(x) \geq g(x) \) for all \( x \in [a, b] \), then \( \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \).

Proof: Let \( h(x) = f(x) - g(x) \). Then \( h(x) \geq 0 \) for all \( x \in [a, b] \), so by part (a) \( \int_a^b h(x) \, dx = \int_a^b (f(x) - g(x)) \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \geq 0 \). Therefore \( \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \).
6. If \( f \) is continuous on \([-\pi, \pi]\), differentiable on \((-\pi, \pi)\), and \( f' \) is integrable on \([-\pi, \pi]\), then for all \( n \in \mathbb{N} \),

\[
\int_{-\pi}^{\pi} f'(x) \sin(nx) \, dx = -n \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx.
\]

**Proof:** Let \( g(x) = \sin(nx) \). Then \( (fg)' = f'g + fg' \) by the product rule. Thus \( f'g = (fg)' - g'f \implies \int f'g = \int [(fg)' - g'f] = fg(b) - fg(a) - \int g'f \) by the Fundamental Theorem of Calculus. Therefore

\[
\int_{-\pi}^{\pi} f'(x) \sin(nx) \, dx = f(\pi) \sin(n\pi) - f(-\pi) \sin(-n\pi) - \int_{-\pi}^{\pi} g'f
\]

(1)

\[
= -\int_{-\pi}^{\pi} n \cos(nx) f(x) \, dx
\]

(2)

\[
= -n \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx
\]

(3)

\[\square\]

**Prove one of the following:**

7. Let \( f \) be differentiable on an interval \( I \). Then if \( f'(x) < 0 \) for all \( x \in I \), then \( f \) is strictly decreasing on \( I \).

**Proof:** Let \( x_1, x_2 \in I \) with \( x_1 < x_2 \). By the Mean Value Theorem there exists \( c \in (x_1, x_2) \) such that

\[ f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \]

This implies \( f(x_2) - f(x_1) = f'(c)(x_2 - x_1) < 0 \) since \( f'(c) < 0 \) and \( x_2 - x_1 > 0 \). Therefore \( f \) is strictly decreasing on \( I \).

\[\square\]

8. Let \( f \) be a monotonic function on \([a, b]\). Then \( f \) is integrable.

**Proof:** We will assume \( f \) is nondecreasing. The other case follows in the same way. Since \( f(a) \leq f(x) \leq f(b) \) for all \( x \in [a, b] \), \( f \) is bounded. Given \( \epsilon > 0 \) there exists \( k > 0 \) such that \( k|f(b) - f(a)| < \epsilon \). Let \( P = \{x_0, x_1, \ldots, x_n\} \) be a partition of \([a, b]\) such that \( \Delta x_i < k \) for all \( i \). Since \( f \) is nondecreasing we have \( m_i = f(x_{i-1}) \) and \( M_i = f(x_i) \). Thus \( U(f, P) - (f, P) = \sum (M_i - m_i) \Delta x_i < k \sum f(x_i) - \sum f(x_{i-1}) = k|f(b) - f(a)| < \epsilon \). Therefore by Theorem 97, \( f \) is integrable.

\[\square\]