Homework 5, Due Friday April 24 2015

Show all the work. Late homework will not be accepted.

Problem.
Consider the homogeneous heat equation problem:
\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad 0 < x < 1, 0 < t \leq T, \\
\end{align*}
\] (1)
subject to the boundary and initial conditions
\[
\begin{align*}
u(0,t) &= 0, u(1,t) = 0, \quad \text{and} \\
u(x,0) &= g(x), \quad 0 < x < 1,
\end{align*}
\]
and approximate it on a uniform grid with the help of the \( \theta \)-scheme:
\[
\begin{align*}
\frac{u_j^{i+1} - u_j^i}{\Delta t} &= (1 - \theta) \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + \theta \frac{u_{i+1}^{j} - 2u_i^j + u_{i-1}^j}{h^2},
\end{align*}
\]
where \( \theta \) is a real parameter, \( 0 \leq \theta \leq 1 \) and \( 0 \leq j \leq M - 1 \) is for the time discretization and \( 1 \leq i \leq N \) is for the space, \( \Delta t = T/M \) and \( h = 1/(N+1) \).

a) Consider case \( \theta = 1 \) (Forward in time scheme) and show that the scheme has the error \( u(x_i, t_j) - u_i^j \) of order \( O(\Delta t) + O(h^2) \) on a smooth solution \( u = u(x,t) \).

b) Investigate theoretically (using von Neumann method as we did on the examples in class) stability of the \( \theta \)-scheme for the heat equation (1), \( 0 \leq \theta \leq 1 \).

c) (Computational part): Please submit codes by e-mail and include the printout and discussion of the results with the theoretical part.

1. Implement schemes in
   a) (Forward in time, \( \theta = 1 \));
   b) Bonus question for extra 35 points (Backward in time, \( \theta = 0 \));
   c) Bonus question for extra 35 points (Crank - Nicolson scheme, \( \theta = 0.5 \)).

Test these schemes using the following set up for the considered homogeneous heat equation problem:
\[
g(x) = \sin(2\pi x).
\]

The exact solution for this test problem is \( u(x,t) = e^{-4\pi^2 t} \sin(2\pi x) \). Consider \( T = 0.1 \).

2. Consider \( h = 1/5, 1/10, 1/20, 1/40, 1/80 \). Choose the appropriate time step \( \Delta t \) for each scheme (and for each \( h \)).

3. Plot the numerical solution (for each scheme and for each choice of \( h \)) versus the exact solution for \( t = 0.02, 0.04 \) and 0.1.

4. Compute maximum error \( \max_i |u(x_i, T) - u_i^M| \) (compute maximum error at the final time \( T = 0.1 \) for each choice of \( h \)) and estimate the rate of convergence.

5. Discuss the results.