Problem 1.
Given the interpolation data (points) \((0, 2), (0.5, 5), (1, 4)\):

(a) Find the function 
\[ f(x) = a_0 + a_1 \cos(\pi x) + a_2 \sin(\pi x), \]
which interpolates the given data

(b) Find the quadratic polynomial interpolating this data
In each case, graph the interpolating function.

Problem 2.
Bound the error (in terms of \(h\) - a positive constant) of the quadratic interpolation to 
\[ f(x) = e^x \]
on \([0, 1]\) with evenly spaced interpolation points \(x_0, x_1 = x_0 + h, x_2 = x_0 + 2h\). Assume that \(x\) such that \(x_0 < x < x_2\)

Problem 3.
(a) Suppose you are given symmetric data:
\[(x_i, y_i), \quad i = -n, -n + 1, ..., n - 1, n,\]
such that
\[x_{-i} = -x_i \text{ and } y_{-i} = -y_i, \quad i = 0, 1, ..., n.\]

What is the required degree of the interpolating polynomial \(p\)? Show that the interpolating polynomial is odd, i.e., \(p(x) = -p(-x)\) for all real numbers \(x\).

(b) Let \(l_i(x)\) are Lagrange basis functions with \(x_0, x_1, ..., x_n\) (as defined in class) with \(n = 2015\). Prove that,
\[ \sum_{i=0}^{2015} l_i(x) = 1 \]
for all \(x\).

Problem 4.
(a) Consider finding a rational function 
\[ p(x) = \frac{a_1 x + b_1}{1 + x^2} \]
that satisfies
\[ p(x_i) = y_i, \quad i = 1, 2, 3 \]
with distinct \(x_1, x_2, x_3\). Does such a function \(p(x)\) exist, or are additional conditions needed to ensure existence and uniqueness of \(p(x)\)?

(b) Let \(x_0, ..., x_n\) be distinct real points, and consider the following interpolation problem. Choose a function
\[ F_n(x) = \sum_{j=0}^{n} c_j e^{jx} \]
such that
\[ F_n(x_i) = y_i \quad i = 0, 1, ..., n \]
with the \( \{ y_i \} \) given data. Show there is a unique choice of \( c_0, \ldots, c_n \).

Problem 5. (Computational assignment. Please submit the codes by e-mail and include the printout of the results (and discussion of the results) with the theoretical part.)

Consider the function \( f(x) = (x^2 + 1)^{-1} \) on the interval \(-5 \leq x \leq 5\). For each \( n \geq 1 \), define \( h = 10/n \), \( x_j = -5 + j \times h \), for \( j = 0, 1, \ldots, n \). Let \( p_n(x) \) be the polynomial of degree \( n \) which interpolates \( f \) at the nodes \( x_0, x_1, \ldots, x_n \). Compute \( p_n \) for \( n = 1, 2, \ldots, 20 \), plot \( f(x) \) and \( p_n(x) \) for each \( n \), and estimate the maximum error \( |f(x) - p_n(x)| \) for \( x \in (-5, 5) \). Discuss what you find.