1. Suppose that $A \ast B \ast C$ and $A \ast C \ast D$.
   (a) (3 pts) Prove that $A, B, C, D$ are four distinct points.

   **Solution:** By B-1, $A, B, C$ are distinct and $A, C, D$ are distinct. The only pair of points that do not appear in both sets is the pair $B, D$. If $B = D$ then substituting $D$ for $B$ in the hypothesis would yield $A \ast D \ast C$ and $A \ast C \ast D$, contradicting B-3. Therefore $B \neq D$.

   (b) (3 pts.) Prove that $A, B, C, D$ are collinear.

   **Solution:** Axiom B-1 and the assumption $A \ast B \ast C$ together imply that the points $A, B, C$ are collinear. Furthermore, the uniqueness part of I-1 guarantees that these points all lie on line $\overline{AC}$. Similarly, Axiom B-1 and $A \ast C \ast D$ together imply that $A, C, D$ are collinear. Again the uniqueness part of I-1 guarantees that these points all lie on $\overline{AC}$. So all four points lie on $\overline{AC}$.

2. Prove Proposition 3.1(ii): For any two distinct points $A$ and $B$, $\overrightarrow{AB} \cup \overrightarrow{BA} = \{\overrightarrow{AB}\}$.

   **Proof.** Step 1 (5 pts.): $\overrightarrow{AB} \cup \overrightarrow{BA} \subset \{\overrightarrow{AB}\}$.

   Let $P \in \overrightarrow{AB} \cup \overrightarrow{BA}$. The proof will be complete once we show that $P \in \{\overrightarrow{AB}\}$. If $P = A$ or $P = B$ then $P$ is on line $\overrightarrow{AB}$ hence in set $\{\overrightarrow{AB}\}$. Now suppose that $P, A, B$ are distinct. If $P \in \overrightarrow{AB}$, then by definition of ray, $P \in \overrightarrow{AB}$ or $A \ast B \ast P$. Having ruled out the possibilities $P = A$ or $P = B$, if $P \in \overrightarrow{AB}$ then $A \ast P \ast B$ by definition of segment. Therefore $A \ast P \ast B$ or $A \ast B \ast P$. In both cases $A, P, B$ all lie on the same line according to B-1; this line is $\overrightarrow{AB}$ by the uniqueness part of I-1. Thus $P \in \{\overrightarrow{AB}\}$. By the same logic, if $P \in \overrightarrow{BA}$ then $B \ast P \ast A$ or $B \ast A \ast P$, and again $P, A, B$ all lie on $\overrightarrow{AB}$, so $P \in \{\overrightarrow{AB}\}$.

   Step 2 (5 pts.): $\{\overrightarrow{AB}\} \subset \overrightarrow{AB} \cup \overrightarrow{BA}$.

   Let $P \in \{\overrightarrow{AB}\}$. The proof will be complete once we show that $P \in \overrightarrow{AB} \cup \overrightarrow{BA}$. If $P = A$ or $P = B$, then $P \in AB$ by definition of segment.
$AB \subset \overrightarrow{AB}$ by definition of ray, and $\overrightarrow{AB} \subset \overrightarrow{AB} \cup \overrightarrow{BA}$ by definition of union, so $P \in \overrightarrow{AB} \cup \overrightarrow{BA}$.

Now suppose that $P, A, B$ are distinct. These points are collinear because we assumed that $P$ lies on $\overrightarrow{AB}$. Thus B-3 gives us $P \ast A \ast B$ or $A \ast P \ast B$ or $A \ast B \ast P$.

- If $P \ast A \ast B$ then $P \in \overrightarrow{BA}$ by definition of ray.
- If $A \ast P \ast B$ then $P \in AB$ by definition of segment, so then $P \in \overrightarrow{AB}$ by definition of ray.
- If $A \ast B \ast P$ then $P \in \overrightarrow{AB}$ by definition of ray.

In all cases $P$ is in $\overrightarrow{AB}$ or $\overrightarrow{BA}$, meaning that $P \in \overrightarrow{AB} \cup \overrightarrow{BA}$.

3. (14 pts.) Let $A$ be an affine plane. Show that the projective completion of $A$, $A^*$ satisfies axioms I1, I2+, I3 and elliptic parallel postulate.

Axiom I2+: For every line $l$ there are at least three distinct points incident with it.

Solution: See pages 59 – 60 in the book. Although few more details could be supplied.