Class #35

More angle sum
GOAL:

- Theorem ♦: In hyperbolic geometry every triangle has angle sum less than 180°.

- Theorem ♦: In Euclidean geometry every triangle has angle sum of 180°.
If you had these, could you prove Theorem ♦ and Theorem ♦?

- Theorem 5.2: If there is a triangle whose angle sum is not 180° then no triangle has angle sum 180°.
- Theorem 5.3: No triangle in neutral geometry can have angle sum greater than 180°.

- Theorem 5.4: If there is a triangle with angle sum 180°, then all triangles have angle sum 180°.
- Theorem 5.5: A rectangle exists iff EPP holds.
- Theorem 5.6: If a rectangle does not exist, there is a triangle with angle sum less than 180°.
- Theorem 5.7: If a rectangle exists, then there are arbitrarily large rectangles. (proved)
- Theorem 5.8: If a rectangle exists, then for any right triangle ΔXYZ (with right angle at X), there is a rectangle □DEFG such that DE>XY and DG>XZ. (proved)
- Theorem 5.9: If a rectangle exists, then every right triangle has angle sum of 180°.

- Theorem 5.10: If every right triangle has angle sum 180°, then every triangle has angle sum 180°.
- Theorem 5.11: If there is a right triangle with angle sum 180°, then a rectangle exists. (proved)
Theorem ♦: In hyperbolic geometry every triangle has angle sum less than 180°.

Every triangle with angle sum less than 180°.

T5.2 and T5.3

There is a triangle with angle sum less than 180°.

T5.6

A rectangle does not exist

Contrapositive of T5.5

In hyperbolic geometry HPP holds $\Rightarrow$ EPP does not hold
Theorem ♦: In Euclidean geometry every triangle has angle sum of 180°.
Theorem 5.4: If there is a triangle with angle sum 180°, then all triangles have angle sum 180°.

Proof:
- If there is a right triangle whose angle sum is 180°, then a rectangle exist (Theorem 5.11)
- This implies that there are arbitrarily large rectangles, and consequently every right triangle has angle sum 180°. (Theorem 5.8, 5.9)
- Therefore all triangles have angle sum 180° (Theorem 5.10).

Remains to be shown: If there is a triangle with angle sum 180°, then there is a right triangle whose angle sum is 180°. How could you do that?
Prove: If there is a triangle with angle sum $180^\circ$, then there is a right triangle whose angle sum is $180^\circ$

\[ \alpha + \beta + \gamma = 180^\circ \]

WLOG $\alpha < 90^\circ$, $\beta < 90^\circ$
Theorem 5.9: If a rectangle exists, then every right triangle has angle sum of 180°.

Strategy: Use 5.8 to put the triangle into a large rectangle. Consider the ΔABC and show that its angle sum is 180°. Then show that this means that each triangle ΔAEC and ΔEBC has to have angle sum 180°. Apply the same reasoning to ΔAEC to show that ΔAEF must have angle sum 180° as well.
Theorem 5.10: If every right triangle has angle sum 180°, then all triangles have angle sum 180°.

WLOG $\alpha<90^\circ$, $\beta<90^\circ$

Apply similar reasoning as in 5.9