EXTERIOR BLOCKS OF NONCROSSING PARTITIONS

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Abstract. This paper defines an exterior block of a noncrossing partition, then gives a formula for the number of noncrossing partitions of the set \([n]\) with \(k\) exterior blocks. Certain identities involving Catalan numbers are derived from this formula.

1. Introduction

A noncrossing partitions is a partition \(\pi\) of the set \([n] := \{1, 2, \ldots, n\}\) such that whenever \(1 \leq a < b < c < d \leq n\) and \(a\) and \(c\) are in the same block of \(\pi\) and \(b\) and \(d\) are in the same block of \(\pi\), then actually \(a, b, c,\) and \(d\) are all in the same block of \(\pi\). The collection of noncrossing partitions of \([n]\) is denoted by \(\text{NC}_n\). We typically write noncrossing partitions using a '/-' to delimit the blocks of the partition and a ',' to delimit the elements within each block. For example, the partition \(\pi = \{\{1, 4, 6\}, \{2, 3\}, \{5\}, \{7\}, \{8, 10\}, \{9\}, \{11, 12\}\} \in \text{NC}_{12}\) is typically written \(\pi = 1, 4, 6/2, 3/5/7/8, 10/9/11, 12\). Notice that we have written the blocks in ascending order of their least element. Noncrossing partitions can be conveniently visualized in their linear representations; that is, we place the nodes \(1, 2, \ldots, n\) in ascending order on a line, and indicate that two elements are in the same block by drawing an arc connecting the two. All the arcs must be drawn in the same half-plane. Figure 1 gives the linear representation of \(1, 4, 6/2, 3/5/7/8, 10/9/11, 12\). We will make use of the linear representation of a noncrossing partition throughout this paper.

2. Exterior Blocks and the Function \(e(n,k)\)

For ease of discussion we give a preliminary definition. Given a block \(B \in \pi\), we will denote the least and greatest elements of \(B\) by \(\text{first}(B)\) and \(\text{last}(B)\), respectively, and will call them the first and last elements of \(B\), respectively.

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Theorem 2.1. Let $\pi \in \text{NC}_n$. A block $B \in \pi$ is an interior block of $\pi$ if there exists a block $C \in \pi$ such that first$(C) < \text{first}(B) \leq \text{last}(B) < \text{last}(C)$. If $B$ is not an interior block, then it is an exterior block of $\pi$.

Intuitively, given a noncrossing partition $\pi$ of $[n]$, an interior block of $\pi$ is one which is nested inside another block in the linear representation of $\pi$. An exterior block of $\pi$ is one which is not nested in any other block. Consider Figure 1 which is the linear representation of $\pi = 1, 4, 6/2, 3/5/7/8, 10/9/11, 12 \in \text{NC}_{12}$. It is easy to see that $\{2, 3\}, \{5\}$ and $\{9\}$ are the interior blocks of $\pi$, while $\{1, 4, 6\}, \{7\}, \{8, 10\}$ and $\{11, 12\}$ are the exterior blocks of $\pi$.

Let $E_{n,k}$ be the subset of $\text{NC}_n$ consisting of all noncrossing partitions of $[n]$ with $k$ exterior blocks and define

$$e(n, k) = \left| E_{n,k} \right|$$

so that $e(n, k)$ counts the number of noncrossing partitions of $[n]$ with $k$ exterior blocks. What sort of function is $e(n, k)$?

Proposition 2.1. $e(n, k) = 0$ whenever $k = 0$ or $k > n$.

Proof. If $k = 0$, we are asking how many noncrossing partitions of $[n]$ have no exterior blocks. It is easy to see that the block containing 1 is an exterior block of any noncrossing partition. Thus $e(n, 0) = 0$. Since any partition of $[n]$ can have at most $n$ blocks, it can have at most $n$ exterior blocks. So if $k > n$, $e(n, k) = 0$. \qed

Theorem 2.1. $e(n, 1) = C_{n-1}$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n$th Catalan number.

Proof. It is easy to see that $e(1, 1) = 1 = C_0$. Assume $n > 1$. It is also easy to see that a noncrossing partition $\pi$ of $[n]$ with one exterior block necessarily has 1 and $n$ in the same block. Call this block $B$ (see Figure 2, where $n = 6, \pi = 1, 4, 6/2, 3/5$ and $B = \{1, 4, 6\}$). The partition

$$\pi' = (\pi \setminus \{B\}) \cup \{B \setminus \{n\}\}$$

is then a noncrossing partition of $[n-1]$ ($\pi'$ is simply $\pi$ with the element $n$ removed; see Figure 3). Define a map $\phi : E_{n,1} \to \text{NC}_{n-1}$ by the above operation $\pi \mapsto \pi'$. The map $\phi$ is clearly invertible, with inverse map $\phi^{-1}$ given by

$$\phi^{-1}(\sigma) = (\sigma \setminus \{A\}) \cup \{A \cup \{n\}\}$$

where $\sigma \in \text{NC}_{n-1}$ and $A$ is the block of $\sigma$ containing the element 1 (see Figures 4 and 5, where $n = 6$, $\sigma = 1, 2/3/4, 5$ and $A = \{1, 2\}$). Therefore $\phi$ is a bijection, proving

$$e(n, 1) = \left| E_{n,1} \right| = \left| \text{NC}_{n-1} \right| = C_{n-1}. \qed$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Linear representation of $\pi = 1, 4, 6/2, 3/5 \in E_{6,1}$}
\end{figure}

Theorem 2.2. $e(n, k) = e(n-1, k-1) + e(n, k + 1)$ for $n \geq 2$ and $k \geq 1$. 

Proof. Clearly $e(n-1, k-1)$ counts the number of noncrossing partitions $\pi$ of $E_{n,k}$ having the singleton $\{n\}$ as a block since $\pi \setminus \{n\} \in E_{n-1,k-1}$. Thus we want to show that $e(n, k+1)$ counts the number of noncrossing partitions of $E_{n,k}$ that do not have $\{n\}$ as a block. Let $E'_{n,k}$ be that set.

It is easy to see that if $k \in [n-1]$ then there exists a noncrossing partition with $k$ exterior blocks whose block containing $n$ is not a singleton. Thus if $E'_{n,k}$ is empty, then necessarily $k \geq n$. But then $e(n, k+1) = 0$ by Proposition 2.1 and we are done.

If $E'_{n,k}$ is not empty, then for any $\pi \in E'_{n,k}$, let $B$ be the block of $\pi$ containing $n$ and let

$$\pi' = (\pi \setminus \{B\}) \cup \{B \setminus \{n\}, \{n\}\}$$

(see Figures 6 and 7, where $n = 6$, $k = 2$, $\pi = 1, 2/3, 4, 6/5$ and $B = \{3, 4, 6\}$). Now $\pi'$ is a noncrossing partition of $[n]$ with more than $k$ exterior blocks. Let $C$ be the block of $\pi'$ just to the right of $B \setminus \{n\}$ in the linear representation of $\pi'$; that is, last$(B \setminus \{n\}) + 1 = \text{first}(C) (B \setminus \{6\} = \{3, 4\}$ and $C = \{5\}$ in Figure 7). Let

$$\pi'' = (\pi \setminus \{C, \{n\}\}) \cup \{C \cup \{n\}\}$$

(see Figure 8). Now $\pi'' \in E_{n,k+1}$. Define a map $\psi : E'_{n,k} \rightarrow E_{n,k+1}$ by the above operation $\pi \mapsto \pi''$. The map $\psi$ is clearly invertible with inverse map $\psi^{-1}$ given by

$$\psi^{-1}(\sigma) = (\sigma \setminus \{A\}) \cup \{D \cup \{n\}, A \setminus \{n\}\}$$

where $\sigma \in E_{n,k+1}$ and $A$ is the block of $\sigma$ containing $n$ and $D$ is the block of $\sigma$ just to the left of $A$ in the linear representation of $\sigma$; that is, last$(D) + 1 = \text{first}(A)$ (see Figures 9 and 10, where $n = 6$, $k = 2$, $\sigma = 1, 2/3, 4, 5, 6$, $A = \{4, 5, 6\}$ and $D = \{3\}$). Therefore, $\psi$ is a bijection and

$$|E'_{n,k}| = |E_{n,k+1}| = e(n, k+1).$$

We have proven the desired recurrence.

This recurrence relations allows us to write out a table of values for $e(n, k)$ (see Figure 11). Notice that the values of the first two columns of this table come

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**Figure 3.** Linear representation of $\pi' = \phi(\pi) = 1, 4/2, 3/5 \in E_{5,2}$

**Figure 4.** Linear representation of $\sigma = 1, 2/3, 4, 5 \in E_{5,3}$

**Figure 5.** Linear representation of $\phi^{-1}(\sigma) = 1, 2, 6/3, 4, 5 \in E_{6,1}$
from Proposition 2.1 and Theorem 2.1, while the rest of the values come from the recurrence relation written as \( e(n, k + 1) = e(n, k) - e(n - 1, k - 1) \).

Catalan numbers abound in this table. Notice that the second and third columns (corresponding to \( k = 1 \) and \( k = 2 \)) contain Catalan numbers. The first column is, of course, given to us by Theorem 2.1. When \( k = 2 \) and \( n \geq 2 \), the recurrence relation plus Proposition 2.1 shows us that

\[
e(n, 2) = e(n, 1) - e(n - 1, 0) = C_{n-1} - 0 = C_{n-1}.
\]

Notice that the \( n \)th row adds up to \( C_n \). This is clear since the sets

\[E_{n,1}, E_{n,2}, \ldots, E_{n,n}\]

partition \( \text{NC}_n \). This fact gives

\[
C_n = |\text{NC}_n| = |\bigcup_{k=1}^n E_{n,k}| = \sum_{k=1}^n |E_{n,k}| = \sum_{k=1}^n e(n, k).
\]

Also notice the strong resemblance of this table with the various formulations of Catalan’s triangle (cf. [1], also sequences A053121, A008315, etc. in [4]). Figure 12 is a typical Catalan triangle. It is also called a Pascal semi-triangle since if \( w(n, k) \) represents the value in the \( n \)th row and \( k \)th column of this table, then for \( n \geq 1 \) and \( k \geq 1 \), \( w(n, k) \) satisfies the recurrence relation

\[
w(n, k) = w(n - 1, k - 1) + w(n - 1, k + 1).
\]

Notice that the diagonals \( w(2n, 0), w(2n - 1, 1), \ldots, w(n, n) \) of this triangle are the rows \( e(n + 1, 1), e(n + 1, 2), \ldots, e(n + 1, n + 1) \) in Figure 11.

**Theorem 2.3.** \( e(n, k) = \frac{k}{n} \binom{2n-k-1}{n-1} \).

**Proof.** It is easy to check that this formula satisfies Proposition 2.1 and Theorems 2.1 and 2.2. \( \square \)
Using the formulation $e(n, k) = \frac{k}{n} \binom{2n - k - 1}{n - 1}$ of Theorem 2.3, we can derive two identities involving Catalan numbers. The first comes by replacing $e(n, k)$ in Equation 2.1 by this formula:

$$C_n = \sum_{k=1}^{n} e(n, k) = \sum_{k=1}^{n} \frac{k}{n} \binom{2n - k - 1}{n - 1}.$$ 

The second identity comes by considering the number of ways the element $n + 1$ can be added to a noncrossing partition $\pi$ of $[n]$ to get a noncrossing partition $\pi'$ of $[n + 1]$. It is clear that if $\pi$ has $k$ exterior blocks, then there are $k + 1$ ways to form a new noncrossing partition $\pi'$: $k$ ways by adding the element $n + 1$ to each of the exterior blocks, and one way by adding the singleton $\{n + 1\}$ to $\pi$. Since there
are $e(n,k)$ noncrossing partitions of $[n]$ with $k$ exterior blocks, there are a total of $(k + 1)e(n,k)$ noncrossing partitions of $[n + 1]$ gotten in this way. Summing these formulae over the possible number of exterior blocks gives

$$C_{n+1} = |NC_{n+1}| = \sum_{k=1}^{n} (k + 1)e(n,k) = \sum_{k=1}^{n} \frac{k(k + 1)}{n} \binom{2n - k - 1}{n - 1}.$$ 

References


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