Applications: The Catenary  If a homogeneous flexible cable or chain is suspended between two fixed points at the same height, it forms a curve called a catenary (Figure 5). Furthermore (see Problem 53), a catenary can be placed in a coordinate system so that its equation takes the form

\[ y = a \cosh \frac{x}{a} \]

**Example 5** Find the length of the catenary \( y = a \cosh(x/a) \) between \( x = -a \) and \( x = a \).

**SOLUTION** The desired length (see Section 5.4) is given by

\[
\int_{-a}^{a} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_{-a}^{a} \sqrt{1 + \sinh^2 \left( \frac{x}{a} \right)} \, dx
\]

\[
= \int_{-a}^{a} \sqrt{\cosh^2 \left( \frac{x}{a} \right)} \, dx
\]

\[
= 2 \int_{0}^{a} \cosh \left( \frac{x}{a} \right) \, dx
\]

\[
= 2a \left[ \sinh \frac{x}{a} \right]_{0}^{a}
\]

\[
= 2a \sinh 1 \approx 2.35a
\]

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**Concepts Review**

1. \( \sinh \) and \( \cosh \) are defined by \( \sinh x = \) _____ and \( \cosh x = \) _____.

2. In **hyperbolic** trigonometry, the identity corresponding to \( \sin^2 x + \cos^2 x = 1 \) is _____.

3. Because of the identity in Question 2, the graph of the parametric equations \( x = \cosh t, y = \sinh t \) is _____.

4. The graph of \( y = a \cosh(x/a) \) is a curve called a _____. This curve is important as a model for _____.

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**Problem Set 6.9**

In Problems 1–12, verify that the given equations are identities.

1. \( e^x = \cosh x + \sinh x \)

2. \( e^{-x} = \cosh x - \sinh x \)

3. \( e^{2x} = \cosh 2x + \sinh 2x \)

4. \( e^{-2x} = \cosh 2x - \sinh 2x \)

5. \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \)

6. \( \sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y \)

7. \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \)

8. \( \cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y \)

9. \( \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \)

10. \( \tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y} \)

11. \( \sinh 2x = 2 \sinh x \cosh x \)

12. \( \cosh 2x = \cosh^2 x + \sinh^2 x \)

In Problems 13–36, find \( D_y y \).

13. \( y = \sinh^2 x \)

14. \( y = \cosh^2 x \)

15. \( y = 5 \sinh^2 x \)

16. \( y = \cosh^3 x \)

17. \( y = \cosh(3x + 1) \)

18. \( y = \sinh(x^2 + x) \)

19. \( y = \ln(\sinh x) \)

20. \( y = \ln(\cosh x) \)

21. \( y = x^2 \cosh x \)

22. \( y = x^2 \sinh x \)

23. \( y = \cosh 3x \sinh x \)

24. \( y = \sinh x \cosh 4x \)

25. \( y = \tanh x \sinh 2x \)

26. \( y = \cosh 4x \sinh x \)

27. \( y = \sinh^{-1}(x^2) \)

28. \( y = \cosh^{-1}(x^3) \)

29. \( y = \tanh^{-1}(2x - 3) \)

30. \( y = \cosh^{-1}(x^3) \)

31. \( y = x \cosh^{-1}(3x) \)

32. \( y = x^2 \sinh^{-1}(x^3) \)

33. \( y = \ln(\cosh^{-1} x) \)

34. \( y = \cosh^{-1}(\cos x) \)
35. \( y = \tanh(\cot x) \)  
36. \( y = \coth^{-1}(\tanh x) \)

37. Find the area of the region bounded by \( y = \cosh 2x, \ y = 0, \ x = 0, \) and \( x = \ln 3. \)

In Problems 38-45, evaluate each integral.

38. \( \int \sinh(3x + 2) \, dx \)
39. \( \int x \cosh(px^2 + 5) \, dx \)
40. \( \int \frac{\cosh \sqrt{x}}{\sqrt{x}} \, dz \)
41. \( \int \frac{\sinh(2x^{1/4})}{\sqrt{x^2}} \, dz \)
42. \( \int e^x \sinh e^x \, dx \)
43. \( \int \cos x \sinh(\sin x) \, dx \)
44. \( \int \tanh x \ln(\cosh x) \, dx \)
45. \( \int x \coth x \ln(\sinh x^2) \, dx \)

46. Find the area of the region bounded by \( y = \cosh 2x, \ y = 0, \ x = -\ln 5, \) and \( x = \ln 5. \)

47. Find the area of the region bounded by \( y = \sinh x, \ y = 0, \ x = \ln 2. \)

48. Find the area of the region bounded by \( y = \tanh x, \ y = 0, \ x = -8, \) and \( x = 8. \)

49. The region bounded by \( y = \cosh x, \ y = 0, \ x = 0, \) and \( x = 1 \) is revolved about the \( x \)-axis. Find the volume of the resulting solid. \( \text{Hint: } \cosh^2 x = (1 + \cosh 2x)/2. \)

50. The region bounded by \( y = \sinh x, \ y = 0, \ x = 0, \) and \( x = \ln 10 \) is revolved about the \( x \)-axis. Find the volume of the resulting solid.

51. The curve \( y = \cosh x, \ 0 \leq x \leq 1, \) is revolved about the \( x \)-axis. Find the area of the resulting surface.

52. The curve \( y = \sinh x, \ 0 \leq x \leq 1, \) is revolved about the \( x \)-axis. Find the area of the resulting surface.

53. To derive the equation of a hanging cable (catenary), we consider the section \( AP \) from the lowest point \( A \) to a general point \( P(x, y) \) (see Figure 6) and imagine the rest of the cable to have been removed.

The forces acting on the cable are
1. \( H = \) horizontal tension pulling at \( A; \)
2. \( T = \) tangential tension pulling at \( P; \)
3. \( W = \delta s = \) weight of \( s \) feet of cable of density \( \delta \) pounds per foot.

To be in equilibrium, the horizontal and vertical components of \( T \) must just balance \( H \) and \( W, \) respectively. Thus, \( T \cos \phi = H \) and \( T \sin \phi = W = \delta s, \) and so
\[
\frac{T \sin \phi}{T \cos \phi} = \tan \phi = \frac{\delta s}{H}
\]

But since \( \tan \phi = \frac{dy}{dx}, \) we get
\[
\frac{dy}{dx} = \frac{\delta s}{H}
\]
and therefore
\[
\frac{d^2 y}{dx^2} = \frac{\delta s}{H} \frac{dx}{dx} = \frac{\delta s}{H} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}
\]

Now show that \( y = a \cosh(x/a) + C \) satisfies this differential equation with \( a = H/\delta. \)

54. Call the graph of \( y = b - a \cosh(x/a) \) an inverted catenary and imagine it to be an arch sitting on the \( x \)-axis. Show that if the width of this arch along the \( x \)-axis is \( 2a \) then each of the following is true.
(a) \( b = a \cosh 1 \approx 1.54308a. \)
(b) The height of the arch is approximately \( 0.54308a. \)
(c) The height of an arch of width \( 48 \) is approximately \( 13. \)

55. A farmer built a large hayshed of length 100 feet and width 48 feet. A cross section has the shape of an inverted catenary (see Problem 54) with equation \( y = 37 - 24 \cosh(x/24). \)
(a) Draw a picture of this shed.
(b) Find the volume of the shed.
(c) Find the surface area of the roof of the shed.

56. Show that \( A = t/2, \) where \( A \) denotes the area in Figure 2 of this section. \( \text{Hint: At some point you will need to use Formula 44 from the back of the book.} \)

57. Demonstrate that for every real number \( r: \)
(a) \( \sinh x + \cosh x)^r = \sinh rx + \cosh rx \)
(b) \( \cosh x - \sinh x)^r = \cosh rx - \sinh rx \)
(c) \( \cos x + i \sin x)^r = \cos rx + i \sin rx \)
(d) \( \cos x - i \sin x)^r = \cos rx - i \sin rx \)

58. The gudermannian of \( t \) is defined by
\[
\text{gd}(t) = \tan^{-1}(\sinh t)
\]
Show that
(a) \( \text{gd} \) is odd and increasing with an inflection point at the origin;
(b) \( \text{gd}(t) = \sinh^{-1}(\tanh t) = \int_0^t \text{sech} u \, du. \)

59. Show that the area under the curve \( y = \cosh t, \) \( 0 \leq t \leq x, \) is numerically equal to its arc length.

60. Find the equation of the Gateway Arch in St. Louis, Missouri, given that it is an inverted catenary (see Problem 54). Assume that it stands on the \( x \)-axis, that it is symmetric with respect to the \( y \)-axis, and that it is 630 feet wide at the base and 630 feet high at the center.

61. Draw the graphs of \( y = \sinh x, \ y = \ln(x + \sqrt{x^2 + 1}), \) and \( y = x \) using the same axes and scaled so that \(-3 \leq x \leq 3 \) and \(-3 \leq y \leq 3.\) What does this demonstrate?