MATH1220: Midterm 3 Practice Problems

The following are practice problems for the second exam.

1. Compute the following limits:
   (a) \( \lim_{x \to 1} \frac{x^2 - 2x + 1}{\sin(\pi x)} = 0 \)
   (b) \( \lim_{x \to 0} \frac{e^x - e^{-x}}{2 \sin x} = 1 \)
   (c) \( \lim_{x \to \infty} \frac{(\ln x)^2}{2^x} = 0 \)
   (d) \( \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e \)

2. Evaluate the following integrals:
   (a) \( \int_{-\infty}^{1} e^{4x} \, dx = \frac{1}{4} e^4 \)
   (b) \( \int_{5}^{\infty} \frac{x}{1 + x^2} \, dx = \infty \)
   (c) \( \int_{-\infty}^{\infty} \frac{dx}{x^2 + 9} = \frac{2\pi}{81} \) Obtain this answer usinga trig substitution.
   (d) \( \int_{0}^{3} \frac{dx}{x^2 - 2x - 3} \) Answer: Diverges
   (e) \( \int_{-4}^{0} \frac{dx}{(x + 3)^2} \) Answer: Diverges

3. Write an explicit formula for the \( n \)-th term of the sequence. Then determine whether the sequence converges or diverges. If it converges, find what number it converges to:
   (a) \( a_1 = 7, \ a_{n+1} = a_n \left( \frac{2}{3} \right) \) Answer: \( a_n = 7 \left( \frac{2}{3} \right)^{n-1} \)
   (b) \(-1, 2, 5, 8, 11, \ldots\) Answer: \( a_n = -4 + 3n \)
   (c) \( 0, \frac{1}{4}, \frac{2}{6}, \frac{3}{8}, \frac{4}{10}, \frac{5}{12}, \frac{6}{14}, \ldots\) Answer: \( a_n = \frac{n-1}{2n} \)

4. Find the limit of the sequence \( a_n = \frac{n}{5n^3 - 2n + 2} - \frac{2}{5} \).

5. Show that the sequence \( a_n = \frac{n}{n + 1} \left( 2 - \frac{1}{n^2} \right) \) converges using the monotone sequence theorem. Answer: \( a_n \) is an increasing sequence because \( \frac{n}{n + 1} \) is increasing as is \( 2 - \frac{1}{n^2} \). Also, \( a_n \) is bounded above by 2 since \( a_n = \frac{n}{n + 1} \left( 2 - \frac{1}{n^2} \right) \leq 1 \cdot 2 = 2 \). Hence it converges.
6. Determine the convergence/divergence of the following series:

(a) \( \sum_{n=1}^{\infty} \frac{n}{n+1} \) Answer: Diverges by the divergence test (the terms don’t go to zero)

(b) \( \sum_{k=1}^{\infty} \left[ 5 \left( \frac{1}{2} \right)^k - 3 \left( \frac{1}{7} \right)^k \right] \) Answer: Converges because it’s a difference of convergent geometric series. It converges to \( \frac{5}{1-1/2} - \frac{3}{1-1/7} \)

(c) \( \sum_{k=1}^{\infty} \frac{2}{(k+2)k} \) Answer: Converges by limit comparison test.

(d) \( \sum_{k=1}^{\infty} \ln(k/(k+1)) \) Answer: Diverges because it’s a collapsing series and the partial sums don’t converge.

(e) \( \sum_{k=1}^{\infty} \frac{3}{2k^2 + 1} \) Answer: Converges by limit comparison test

(f) \( \sum_{k=1}^{\infty} \frac{1000k^2}{1+k^3} \) Answer: Diverges by limit comparison test.

(g) \( \sum_{k=1}^{\infty} k \sin(1/k) \) Answer: Diverges by the divergence test. \( \lim_{k \to \infty} k \sin(1/k) = \lim_{k \to \infty} \frac{\sin(1/k)}{1/k} = \lim_{x \to 0} \frac{\sin x}{x} \).

(h) \( \sum_{k=1}^{\infty} \frac{\sqrt[3]{3n^4} + 3}{n^2} \) Answer: Converges by limit comparison test.

(i) \( \sum_{k=1}^{\infty} \frac{3^k + k}{k!} \) Answer: Converges by ratio test.

(j) \( \sum_{k=1}^{\infty} \frac{\ln k}{2^k} \) Answer: Converges by ratio test.

(k) \( \sum_{k=1}^{\infty} \frac{4^{2n}}{n!} \) Answer: Converges by ratio test.

(l) \( \frac{\ln 2}{2^2} + \frac{\ln 3}{3^3} + \frac{\ln 4}{4^4} + \frac{\ln 5}{5^5} + \ldots \) Answer: Converges by ratio test. Computing the limit is difficult because it involves several tricks.

7. Determine whether each of the following is absolutely convergent, conditionally convergent, or divergent:

(a) \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \) Answer: Conditionally convergent by alternating series test and absolute ratio test.
(b) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{e^k}$ Answer: Absolutely convergent by absolute ratio test.

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n}$ Answer: Divergent by alternating series test (terms don’t go to zero).