Homework # 10

10.4 # 9: (3 pts, p. 724) Find the velocity, acceleration, and speed of a particle with the given position function.

\[ r(t) = \sqrt{2} t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k} \]

The velocity of the particle is simply \( r'(t) \):

\[ v(t) = r'(t) = \sqrt{2} \mathbf{i} + e^t \mathbf{j} - e^{-t} \mathbf{k} \]

The acceleration of the particle is \( r''(t) \):

\[ a(t) = r''(t) = e^t \mathbf{j} + e^{-t} \mathbf{k} \]

The speed of the particle is \(|r'(t)|\):

\[ s(t) = |r'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} \]
\[ = \sqrt{2 + e^{2t} + e^{-2t}} \]
\[ = \sqrt{e^{2t} + 2 + e^{-2t}} \]
\[ = \sqrt{(e^t + e^{-t})^2} \]
\[ = e^t + e^{-t} \]

10.4 # 13: (3 pts, p. 724) Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

\[ a(t) = \mathbf{i} + 2 \mathbf{j}, \quad v(0) = \mathbf{k}, \quad r(0) = \mathbf{i} \]

The velocity of the particle is the integral of the acceleration adjusted so that the value of \( v(0) = \mathbf{k} \). Note that the vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) are treated as constants in the process of integration:

\[ v(t) = \int v(0) \, dt \]
\[ = \int (\mathbf{i} + 2 \mathbf{j}) \, dt \]
\[ = t \mathbf{i} + 2t \mathbf{j} + \mathbf{C} \]
With \( \mathbf{C} \) an arbitrary constant vector

\[ v(0) = \mathbf{k} \]

The initial condition

\( (0)\mathbf{i} + 2 \cdot (0)\mathbf{j} + \mathbf{C} = \mathbf{k} \)
\[ \mathbf{C} = \mathbf{k} \]
Value of the arbitrary constant vector = \( v(0) \)

\[ v(t) = ti + 2tj + k \]
Full value of \( v(t) \)

The value of the position vector is the integral of the velocity vector again adjusted for the
the initial position:

$$r(t) = \int v(t) \, dt$$

$$= \int (t \mathbf{i} + 2t \mathbf{j} + \mathbf{k}) \, dt$$

$$= \frac{1}{2} t^2 \mathbf{i} + t^2 \mathbf{j} + t \mathbf{k} + \mathbf{C} \quad \text{With } \mathbf{C} \text{ and arbitrary constant vector}$$

$$r(0) = \mathbf{i} \quad \text{The initial condition}$$

$$\frac{1}{2} \cdot (0)^2 \mathbf{i} + (0)^2 \mathbf{j} + 0 \mathbf{k} + \mathbf{C} = \mathbf{i}$$

$$\mathbf{C} = \mathbf{i} \quad \text{Value of the arbitrary constant vector } = r(0)$$

$$r(t) = \left( \frac{1}{2} t^2 + 1 \right) \mathbf{i} + t^2 \mathbf{j} + tk \quad \text{Full value of } r(0)$$

10.4 # 21: (4 pts, p. 725) A projectile is fired with an initial speed of 200 m/s and angle of elevation 60°. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

This is a mathematical approach to the problem, in a physics class, the same answers could be found by modeling the trajectory as a parabola and using formulas for that model.

To find the trajectory of the projectile, first assume that there is no air resistance. Then the only force on the projectile is that of gravity. The Earth is assumed to flat and level, to that the point of impact will have the same vertical level as the point of the initial firing of the projectile. The acceleration of the projectile is $a(t) = -g \mathbf{k}$. Assume that the projectile is fired in the direction of the $x$ axis so that its initial velocity is $v(0) = 200 \cos 60^\circ \mathbf{i} + 200 \sin 60^\circ \mathbf{k}$. Also, to make calculations easier, assume that the initial position is $r = 0$. Now the velocity is the integral of the acceleration adjusted for the initial velocity:

$$v(t) = \int a(t) \, dt$$

$$= \int ( -g \mathbf{k}) \, dt$$

$$= -gt \mathbf{k} + \mathbf{C} \quad \text{With } \mathbf{C} \text{ an arbitrary constant vector}$$

$$v(0) = 200 \cos 60^\circ \mathbf{i} + 200 \sin 60^\circ \mathbf{k} \quad \text{The initial condition}$$

$$-g(0) \mathbf{k} = 200 \cos 60^\circ \mathbf{i} + 200 \sin 60^\circ \mathbf{k}$$

$$\mathbf{C} = 200 \cos 60^\circ \mathbf{i} + 200 \sin 60^\circ \mathbf{k}$$

$$v(t) = 200 \cos 60^\circ \mathbf{i} + ( -gt + 200 \sin 60^\circ ) \mathbf{k} \quad \text{The velocity}$$

The position of the projectile can be found as the integral of the velocity:

$$r(t) = \int v(t) \, dt$$

$$= \int \left[ 200 \cos 60^\circ \mathbf{i} + ( -gt + 200 \sin 60^\circ ) \mathbf{k} \right] \, dt$$

$$= 200t \cos 60^\circ \mathbf{i} + ( -\frac{1}{2} gt^2 + 200t \sin 60^\circ ) \mathbf{k} + \mathbf{C}$$

$$r(0) = 0$$

$$\mathbf{C} = 0$$

$$r(t) = 200t \cos 60^\circ \mathbf{i} + ( -\frac{1}{2} gt^2 + 200t \sin 60^\circ ) \mathbf{k}$$
The range of the projectile can be found by finding when the projectile returns to a height of 0 meters. Substituting that time into the $i$ component of the trajectory, or $200t\cos 60^\circ$, will give the distance that the projectile traveled horizontally. The height of the projectile is the $k$ component of the projectile’s trajectory or $\frac{1}{2}gt^2 + 200t\sin 60^\circ$. Solving for when this value is zero:

$$-\frac{1}{2}gt^2 + 200t\sin 60^\circ = 0$$

$$t = \begin{cases} 0 & \text{When the projectile is initially fired} \\ \frac{400\sin 60^\circ}{g} & \text{When the projectile returns to the Earth} \end{cases}$$

range = $200\left(\frac{400\sin 60^\circ}{g}\right)\cos 60^\circ$

$$= \frac{8000\sin 60^\circ \cos 60^\circ}{g}$$

$$\approx 3531.1943 \text{ meters} \quad \text{Using } g = 9.81 \text{ m/s}^2$$

To find the maximum height, use the $k$ component of the trajectory and find its maximum; takes its derivative, find when it is equal to zero, and substitute that time back into the $k$ component:

$$\frac{d}{dt} \left( -\frac{1}{2}gt^2 + 200t\sin 60^\circ \right) = -gt + 200\sin 60^\circ = 0$$

$$t = \frac{200\sin 60^\circ}{g}$$

maximum height = $-\frac{1}{2}g\left(\frac{200\sin 60^\circ}{g}\right)^2 + 200\left(\frac{200\sin 60^\circ}{g}\right)\sin 60^\circ$

$$= \frac{20000\sin^2 60^\circ}{g}$$

$$\approx 1529.0520 \text{ meters} \quad \text{Using } g = 9.81 \text{ m/s}^2$$

The speed at impact is the value of $|\vec{v}(t)|$ at the time the projectile returns to Earth. Without air resistance, the parabola which the projectile travels is symmetric and the speed is the same as the initial speed. To calculate this from a mathematical approach, use the time at which the projectile returns to Earth which was calculated above as $\frac{400\sin 60^\circ}{g}$ to calculate the value of $|\vec{v}(t)|$:

$$|\vec{v}(t)| = \sqrt{(200\cos 60^\circ)^2 + \left[-g\left(\frac{400\sin 60^\circ}{g}\right) + 200\sin 60^\circ\right]^2}$$

$$= \sqrt{(200\cos 60^\circ)^2 + \left(-200\sin 60^\circ\right)^2}$$

$$= \sqrt{40000\cos^2 60^\circ + 40000\sin^2 60^\circ}$$

$$= \sqrt{40000} = 200 \text{ m/s}$$