1. Some examples

Example 16.1 (Example 14.2, continued). We find that

\[ E(XY) = \left( 1 \times 1 \times \frac{2}{36} \right) = \frac{2}{36}. \]

Also,

\[ EX = EY = \left( 1 \times \frac{10}{36} \right) + \left( 2 \times \frac{1}{36} \right) = \frac{12}{36}. \]

Therefore,

\[ \text{Cov}(X, Y) = \frac{2}{36} - \left( \frac{12}{36} \times \frac{12}{36} \right) = -\frac{72}{1296} = -\frac{1}{18}. \]

The correlation between \( X \) and \( Y \) is the quantity,

\[ \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}. \quad (14) \]

Example 16.2 (Example 14.2, continued). Note that

\[ E(X^2) = E(Y^2) = \left( 1^2 \times \frac{10}{36} \right) + \left( 2^2 \times \frac{1}{36} \right) = \frac{14}{36}. \]

Therefore,

\[ \text{Var}(X) = \text{Var}(Y) = \frac{14}{36} - \left( \frac{12}{36} \right)^2 = \frac{360}{1296} = \frac{5}{13}. \]

Therefore, the correlation between \( X \) and \( Y \) is

\[ \rho(X, Y) = -\frac{1/18}{\sqrt{\left( \frac{5}{13} \right) \left( \frac{5}{13} \right)}} = -\frac{13}{90}. \]
2. Correlation and independence

The following is a variant of the Cauchy–Schwarz inequality. I will not prove it, but it would be nice to know the following.

**Theorem 16.3.** If $E(X^2)$ and $E(Y^2)$ are finite, then $-1 \leq \rho(X, Y) \leq 1$.

We say that $X$ and $Y$ are uncorrelated if $\rho(X, Y) = 0$; equivalently, if $\text{Cov}(X, Y) = 0$. A significant property of uncorrelated random variables is that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$; see Theorem 15.4(2).

**Theorem 16.4.** If $X$ and $Y$ are independent [with joint mass function $f$], then they are uncorrelated.

**Proof.** It suffices to prove that $E(XY) = E(X)E(Y)$. But

$$E(XY) = \sum_x \sum_y xyf(x, y) = \sum_x \sum_y x y f_X(x)f_Y(y)$$

$$= \sum_x x f_X(x) \sum_y y f_Y(y) = E(X)E(Y),$$

as planned. \[\square\]

**Example 16.5** (A counter example). Sadly, it is only too common that people some times think that the converse to Theorem 16.4 is also true. So let us dispel this with a counterexample: Let $Y$ and $Z$ be two independent random variables such that $Z = \pm 1$ with probability $1/2$ each; and $Y = 1$ or $2$ with probability $1/2$ each. Define $X = YZ$. Then, I claim that $X$ and $Y$ are uncorrelated but not independent.

First, note that $X = \pm 1$ and $\pm 2$, with probability $1/4$ each. Therefore, $E(X) = 0$. Also, $XY = Y^2Z = \pm 1$ and $\pm 4$ with probability $1/4$ each. Therefore, again, $E(XY') = 0$. It follows that

$$\text{Cov}(X, Y) = \text{E}(XY') - \underbrace{E(X)}_{0} \underbrace{E(Y)}_{0} = 0.$$  

Thus, $X$ and $Y$ are uncorrelated. But they are not independent. Intuitively speaking, this is clear because $|X| = Y$. Here is one way to logically justify our claim:

$$P\{X = 1, Y = 2\} = 0 \neq \frac{1}{8} = P\{X = 1\}P\{Y = 2\}.$$  

**Example 16.6** (Binomials). Let $X = \text{Bin}(n, p)$ denote the total number of successes in $n$ independent success/failure trials, where $P\{\text{success per trial}\} = p$.
3. The law of large numbers

Theorem 16.7. Suppose $X_1, X_2, \ldots, X_n$ are independent, all with the same mean $\mu$ and variance $\sigma^2 < \infty$. Then for all $\epsilon > 0$, however small,

$$\lim_{n \to \infty} P \left\{ \left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \epsilon \right\} = 0. \quad (15)$$

Lemma 16.8. Suppose $X_1, X_2, \ldots, X_n$ are independent, all with the same mean $\mu$ and variance $\sigma^2 < \infty$. Then:

$$E \left( \frac{X_1 + \cdots + X_n}{n} \right) = \mu$$

$$\text{Var} \left( \frac{X_1 + \cdots + X_n}{n} \right) = \frac{\sigma^2}{n}.$$ 

Proof of Theorem 16.7. Recall Chebyshev’s inequality: For all random variables $Z$ with $E(Z^2) < \infty$, and all $\epsilon > 0$,

$$P \left\{ |Z - EZ| \geq \epsilon \right\} \leq \frac{\text{Var}(Z)}{\epsilon^2}.$$ 

We apply this with $Z = (X_1 + \cdots + X_n)/n$, and then use Lemma 16.8 to find that for all $\epsilon > 0$,

$$P \left\{ \left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \epsilon \right\} \leq \frac{\sigma^2}{n\epsilon^2}.$$ 

Let $n \not\to \infty$ to finish. \qed