1. A continuous random variable $X$ has density function $f$ given by the following:

$$f(x) = \begin{cases} Ce^{-x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Compute $C$.

**Solution:** $1 = C \int_{0}^{\infty} e^{-x} \, dx = C$, so $C = 1$.

(b) Find $P\{X > 10\}$.

**Solution:** $P\{X > 10\} = \int_{10}^{\infty} e^{-x} \, dx = e^{-10} \approx 0.000045$.

2. A salesman has scheduled two appointments to see encyclopedias. His first appointment leads to a sale with probability 0.3, and his second with probability 0.6 independently of the outcome of the first appointment. Any sale made is equally likely to be either for the deluxe model which costs $1,000, or the standard model which costs $500. Let $X$ denote the total value of all the salesman’s sales. Compute $EX$.

**Solution:** For $i = 1, 2$ consider the events $S_i := \{\text{sale on the } i\text{th appointment}\}$. We know that $S_1$ and $S_2$ are independent, $P(S_1) = 0.3$, and $P(S_2) = 0.6$. Let $D_i := \{\text{deluxe on } i\text{th}\}$, also. We know that $P(D_i | S_i) = P(D_i | S_1) = P(D_i | S_2) = 1/2$. Consequently, $P(S_i \cap D_i) = P(S_i)/2$ and $P(S_i \cap D_i^c) = P(S_i)/2$.

The possible values of $X$ are:

- 2000 dollars. In this case, we have $P\{X = 2000\} = P(S_1 \cap D_1)P(S_2 \cap D_2) = \frac{0.3}{2} \times \frac{0.6}{2} = 0.045$;

- 1500 dollars. In this case, we have

$$P\{X = 1500\} = P(S_1 \cap D_1)P(S_2 \cap D_2^c) + P(S_1 \cap D_2)P(S_2 \cap D_2)$$

$$= \left(\frac{0.3}{2} \times \frac{0.6}{2}\right) + \left(\frac{0.3}{2} \times \frac{0.6}{2}\right) = 0.09.$$

- 1000 dollars. In this case, we have

$$P\{X = 1000\} = P(S_1 \cap D_1)P(S_2^c)P(S_2 \cap D_2) + P(S_1 \cap D_1^c)P(S_2 \cap D_2) + P(S_1 \cap D_2^c)P(S_2 \cap D_2^c)$$

$$= \left(\frac{0.3}{2} \times 0.4\right) + \left(\frac{0.6}{2} \times 0.6\right) + \left(\frac{0.3}{2} + \frac{0.6}{2}\right) = 0.315.$$

- 500 dollars. In this case, we have

$$P\{X = 500\} = P(S_1 \cap D_1^c)P(S_2^c)P(S_2 \cap D_2^c) + P(S_1 \cap D_1)P(S_2 \cap D_2)$$

$$= \left(\frac{0.3}{2} \times 0.4\right) + \left(\frac{0.7}{2} \times \frac{0.6}{2}\right) = 0.27.$$

- 0 dollars. In this case, we have $P\{X = 500\} = P(S_1^c)P(S_2^c) = 0.7 \times 0.4 = 0.28$.

Therefore,

$$EX = (2000 \times 0.045) + (1500 \times 0.09) + (1000 \times 0.315) + (500 \times 0.27) = 675 \text{ dollars}.$$

3. Suppose $X$ is a uniform $(0, 1)$ random variable. Then compute $E[X^n]$ for any integer $n \geq 1$.

**Solution:** Evidently, $E[X^n] = \int_{0}^{1} x^n \, dx = 1/(n + 1)$. 

4. Suppose $X$ is normally distributed with $\mu = 1$ and $\sigma^2 = 4$. Then compute $P\{X \geq 0\}$.
   
   **Solution:** Let $\Phi$ denote the standard-normal distribution function. Then, by standardization,
   
   $$P\{X < 0\} = \Phi\left(\frac{0 - 1}{2}\right) = \Phi(-0.5) = 1 - \Phi(0.5).$$
   
   Therefore, $P\{X \geq 0\} = \Phi(0.5) \approx 0.6915$, thanks to the normal table.

5. Let $X$ be a random variable with density function

   $$f(x) = \begin{cases} \frac{3}{x^4}, & \text{if } x > 1, \\ 0, & \text{otherwise.} \end{cases}$$

   **Compute the mean and variance of $X$.**

   **Solution:** We have

   $$EX = \int_{1}^{\infty} \frac{3}{x^3} \, dx = \frac{3}{2},$$
   $$E[X^2] = \int_{1}^{\infty} \frac{3}{x^2} \, dx = 3,$$
   $$\text{Var}X = E[X^2] - (EX)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}. $$