Problems:

1. Suppose $Y$ is uniformly distributed on $(0, 5)$. What is the probability that the roots of the equation $4x^2 + 4xY + Y + 2 = 0$ are both real?

Solution: The two roots of the quadratic are:

$$x = \frac{-4 \pm \sqrt{16Y^2 - 16(Y + 2)}}{8} = \frac{1}{2} \left[-1 \pm \sqrt{Y^2 - Y - 2}\right].$$

The roots are real if and only if $Y^2 - Y - 2 \geq 0$. Consider next the quadratic equation $y^2 - y - 2 = 0$. The solutions are

$$y = \frac{1 \pm \sqrt{1 + 8}}{2} = \frac{1 \pm 3}{2} = -1 \text{ or } 2.$$

This means that $y^2 - y - 2 = (y + 1)(y - 2)$, which can be check directly too. Consequently, $Y^2 - Y - 2 \geq 0$ if and only if $(Y+1)(Y-2) \geq 0$. Because $0 \leq Y \leq 5$, this means that $Y^2 - Y - 2 \geq 0$ if and only if $Y - 2 \geq 0$. Thus, the probability of real roots is $P\{Y \geq 2\} = 3/5$.

2. Two fair dice are rolled. Find the joint mass function of $(X, Y)$ when:
   (a) $X$ is the maximum (i.e., largest) of the values of the two dice, and $Y$ is the sum of the values of the two dice;
   Solution: The possible values of $(X, Y)$ are $(1, 2), (2, 3), (2, 4), (3, 4), \ldots, (3, 6), (4, 5), \ldots, (4, 8), (5, 6), \ldots, (5, 10), (6, 7), \ldots, (6, 12)$. The probabilities are:
   - $p(1, 2) = 1/36$ (one and one);
   - $p(2, 3) = 2/36$ (a two and a one);
   - $p(3, 4) = p(3, 5) = 2/36$, and $p(3, 6) = 1/36$;
   - $p(4, 5) = p(4, 6) = p(4, 7) = 2/36$ and $p(4, 8) = 1/36$;
   - $p(5, 6) = p(5, 7) = p(5, 8) = p(5, 9) = 2/36$ and $p(5, 5) = 1/36$;
   - $p(6, 7) = p(6, 8) = p(6, 9) = p(6, 10) = p(6, 11) = 2/36$ and $p(6, 12) = 1/36$.

   (b) $X$ is the value of the first die and $Y$ is the maximum of the values of the two dice;
   Solution: This is done similarly to the previous one. The probabilities are:
   - $p(1, 1) = 1/36; p(2, 2) = 2/36; p(3, 3) = 3/36; \ldots; p(6, 6) = 6/36$;
   - $p(1, 2) = p(1, 3) = p(1, 4) = p(1, 5) = p(1, 6) = 1/36$;
   - $p(2, 3) = p(2, 4) = p(2, 5) = p(2, 6) = 1/36$;
   - $p(3, 4) = p(3, 5) = p(3, 6) = 1/36$;
   - $p(4, 5) = p(4, 6) = 1/36$;
   - $p(5, 6) = 1/36$;
(c) \(X\) is the minimum (i.e., smallest) of the values of the two dice, and \(Y\) is the maximum of the two values.

Solution: This is done similarly to the previous one. The probabilities are:

- \(p(1, 1) = 1/36\) and \(p(1, 2) = p(1, 3) = p(1, 4) = p(1, 5) = p(1, 6) = 2/36;\)
- \(p(2, 2) = 1/36\) and \(p(2, 3) = p(2, 4) = p(2, 5) = p(2, 6) = 2/36;\)
- \(p(3, 3) = 1/36\) and \(p(3, 4) = p(3, 5) = p(3, 6) = 2/36;\)
- \(p(4, 4) = 1/36\) and \(p(4, 5) = p(4, 6) = 2/36;\)
- \(p(5, 5) = 1/36\) and \(p(5, 6) = 2/36;\)
- \(p(6, 6) = 1/36.\)

3. Consider a sequence of independent Bernoulli trials, each of which is a success with probability \(p\). Let \(X_1\) denote the number of failures preceding the first success, and let \(X_2\) be the number of failures between the first two successes. Find the joint mass function of \((X_1, X_2)\).

Solution: The possible values are all two-dimensional integers of the form \((i, j)\), where \(i, j \geq 0\). Thus, we have

\[
p(i, j) = P(F_1 \cap \cdots \cap F_i \cap S_{i+1} \cap F_{i+2} \cap \cdots \cap F_{i+j+2}) = p(1-p)^{i+j}, \quad i, j = 0, 1, 2, \ldots,
\]

where \(S_i := \{\text{success at the } i\text{th}\}\) and \(F_i := S_i^c\).

4. The joint density function of \((X, Y)\) is given by

\[
f(x, y) = \begin{cases} c(y^2 - x^2)e^{-y}, & \text{if } -y \leq x \leq y \text{ and } 0 < y < \infty, \\ 0, & \text{otherwise.} \end{cases}
\]

(a) Find \(c\).

Solution: Solve for \(c\) as usual:

\[
1 = c \int_0^\infty \int_{-y}^y (y^2 - x^2)e^{-y} \, dx \, dy
= c \int_0^\infty e^{-y} \left( \int_{-y}^y (y^2 - x^2) \, dx \right) \, dy
= c \int_0^\infty e^{-y} \left( 2y^3 - \frac{1}{3} x^3 \right|_{-y}^y \right) \, dy
= 4c \int_0^\infty y^3 e^{-y} \, dy = 4c \frac{1}{3} \Gamma(4) = 4c \frac{1}{3} \times 3! = 8c.
\]

Therefore, \(c = 1/8.\)

(b) Find the (marginal) density functions of \(X\) and \(Y\) respectively.

Solution: Integrate each variable separately: First, suppose \(x > 0\) and note that \(f_X(x) = (1/8) \int_x^\infty (y^2 - x^2)e^{-y} \, dy\). Split the integrals and compute by parts to find that \(\int_x^\infty y^2 e^{-y} \, dy = x^2 e^{-x} + 2 \int_x^\infty y e^{-y} \, dy\). Also, \(\int_x^\infty e^{-y} \, dy = e^{-x}\). Therefore,

\[
f_X(x) = \frac{1}{4} \int_x^\infty y e^{-y} \, dy = \frac{1}{4} (x + 1)e^{-x}, \quad \text{for } x > 0.
\]
If \( x < 0 \), then \( f_X(x) = f_X(-x) \), by symmetry.

Next we compute \( f_Y \): For all \( y > 0 \),

\[
f_Y(y) = \frac{1}{8} \int_{-y}^{y} (y^2 - x^2)e^{-y} \, dx = \frac{1}{8} \left[ 2y^3e^{-y} - e^{-y} \int_{-y}^{y} x^2 \, dx \right] \\
= \frac{1}{8} \left[ 2y^3e^{-y} - \frac{2}{3}y^3e^{-y} \right] = \frac{1}{6}y^3e^{-y}.
\]

If \( y < 0 \) then \( f_Y(y) = 0 \).

(c) Find \( E(X) \).

Solution: Because \( f_X \) is symmetric, \( E(X) = 0 \).

(d) Find \( P\{X > Y\} \).

Solution: Zero because \( P\{X > Y\} = \int \int_{x>y} f(x, y) \, dx \, dy \), and \( f(x, y) = \) if \( x > y \).

5. The (joint) density function of \((X, Y)\) is given by

\[
f(x, y) = \begin{cases} 
\frac{1}{2}e^{-(x+y)}, & \text{if } 0 \leq x < \infty, \text{ and } 0 \leq y < \infty, \\
0, & \text{otherwise}.
\end{cases}
\]

Find: (a) \( P\{X < Y\} \); and (b) \( P\{X < a\} \) for all real numbers \( a \).

Solution: First of all, note that

\[
f(x, y) = f_X(x) \cdot f_Y(y),
\]

where

\[
f_X(x) = \begin{cases} 
e^{-x}, & \text{if } x > 0, \\
0, & \text{otherwise},
\end{cases}, \quad \text{and} \quad f_Y(y) = \begin{cases} e^{-y}, & \text{if } y > 0, \\
0, & \text{otherwise}.
\end{cases}
\]

Therefore, \( X \) and \( Y \) are independent; both are exponentially distributed with mean one. In particular, \( P\{X < Y\} = P\{X > Y\} \). Since \( P\{X = Y\} = 0 \), it follows then that \( P\{X > Y\} = 1/2 \). [Do this by integration as well!] On the other hand,

\[
P\{X < a\} = \int_{0}^{a} e^{-x} \, dx = 1 - e^{-a}.
\]

if \( a > 0 \), and \( P\{X < a\} = 0 \) if \( a < 0 \).