LECTURE 5: THE CRITICAL PERCOLATION PROBABILITY
FOR BOND PERCOLATION

Recall that, in percolation, each edge in $Z^d$ is open or closed with probability $p$ or $(1 - p)$, and the status of all edges are independent from one another. In (4.1, Lecture 4) we showed that there exists a critical probability $p_c$ (sometimes written as $p_c(Z^d)$ to emphasize the lattice in question), such that for all $p > p_c$, there is percolation (i.e., with positive probability, there exists an infinite connected open path from the origin), and for $p < p_c$, there is no percolation. However, this statement is completely vacuous if the numerical value of $p_c$ were trivial in the sense that $p_c$ were 0 or 1. In this lecture, we will show that this is not the case. In fact, we will show that in all dimensions $d \geq 2$,

\begin{equation}
\frac{1}{C(d)} \leq p_c(Z^d) \leq 1 - \frac{1}{C(d)},
\end{equation}

where $C(d)$ is the connectivity constant of $Z^d$; see (§4.2, lecture 3).

(0.2) **Concrete Bounds on** $p_c(Z^d)$. Since that $d \leq C(d) \leq (2d)$ (§4.2, lecture 3), then it follows from (0.1) above that $\frac{1}{2d} \leq p_c(Z^d) \leq 1 - \frac{1}{2d}$. This can be easily improved upon, since by §4.9 of lecture 4, $C(d) \leq (2d - 1)$, so that $\frac{1}{2d - 1} \leq p_c(Z^d) \leq 1 - \frac{1}{2d - 1}$. In particular, $p_c(Z^d)$ is strictly between 0 and 1, which is the desired claim. ♠

(0.3) **The Planar Case.** The planar case deserves special mention: The previous bounds show that $p_c(Z^2)$ is between $\frac{1}{3}$ and $\frac{2}{3}$. In fact, it has been shown that

- **a.** $p_c(Z^2) = \frac{1}{2}$ (Harris and Kesten);
- **b.** If $p = p_c(Z^2)$, then there is no percolation (Bezuidenhout and Grimmett). ♠

§1. THE LOWER BOUND IN (0.1).

We first verify the lower bound of (0.1) on $p_c$. Note that showing $p_c \geq \frac{1}{C(d)}$ amounts to showing that whenever $p < \frac{1}{C(d)}$, then $P\{\text{percolation}\} = 0$.

First note that the chance that any self-avoiding path $\pi$ of length $n$ is open is $p^n$. Therefore,

\begin{equation}
E \{\# \text{ of self-avoiding paths of length } n\} = E \left[ \sum_\pi 1\{\pi \text{ is open}\} \right] = \sum_\pi P\{\pi \text{ is open}\} = \sum_\pi p^n,
\end{equation}

where $\sum_\pi$ denotes the summation over all self-avoiding paths of length $n$, and $1\{\cdots\} := 1\{\cdot\cdots\}$ is the indicator of $\{\cdot\cdots\}$. Since there are $\chi_n$ many self-avoiding paths of length $n$,

\begin{equation}
E \{\# \text{ of self-avoiding paths of length } n\} \leq \chi_n p^n.
\end{equation}
But \( \chi_n \approx \{ C(d) \}^n \), where

\[
(1.3) \quad a_n \approx b_n \quad \text{mean} \quad \lim_{n \to \infty} \frac{\log a_n}{\log b_n} = 1.
\]

This means that as soon as \( p < \frac{1}{C(d)} \), then

\[
(1.4) \quad E\{ \# \text{ of self-avoiding paths of length } n \} \to 0, \quad (n \to \infty).
\]

(Why? Be sure that you understand this!) But for any \( n \),

\[
(1.5) \quad P\{ \text{percolation} \} \leq P\{ \# \text{ of self-avoiding paths of length } n \geq 1 \}
\]

\[
\leq E\{ \# \text{ of self-avoiding paths of length } n \},
\]

thanks to Markov’s inequality (§4.1, lecture 2). Since \( P\{ \text{percolation} \} \) is independent of \( n \), (1.3) shows that it must be zero as long as \( p < \frac{1}{C(d)} \). This shows that \( p_c \geq C(d) \), which is the desired result.

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§2. THE UPPER BOUND IN (0.1).

Now we want to prove the second inequality in (0.1). That is, we wish to show that if \( p > 1 - \frac{1}{C(d)} \), then \( P\{ \text{percolation} \} = 0 \). This is trickier to do, since we have to produce an open path or an algorithm for producing such a path, and this is a tall order. Instead, let us prove the (logically equivalent) converse to the bound that we are trying to prove. Namely, we show that if \( P\{ \text{percolation} \} = 0 \), then \( p \leq 1 - \frac{1}{C(d)} \). For this, we need to briefly study a notion of duality for percolation, and one for graphs. From now on, we will only work with \( \mathbb{Z}^2 \); once you understand this case, you can extend the argument to get the upper bound in (0.1) for any \( d \geq 2 \).

(2.1) The Dual Lattice. Briefly speaking, the dual lattice \( \mathbb{Z}^* \) of \( \mathbb{Z}^2 \) is the lattice

\[
(2.2) \quad \mathbb{Z}^* := \mathbb{Z}^2 + \left( \frac{1}{2}, \frac{1}{2} \right).
\]

At this point, some of you may (and should) be asking yourselves, “What does it mean to sum a set and a point?” In general, \( A + x \) is short-hand for the set \( \{ y + x ; y \in A \} \). That is, \( A + x \) is \( A \) shifted by \( x \). Consequently, the dual lattice \( \mathbb{Z}^* \) is the lattice \( \mathbb{Z}^2 \) shifted by \( (0.5, 0.5) \). Pictorially speaking, the dual lattice \( \mathbb{Z}^* \) looks just like \( \mathbb{Z}^2 \), except that its origin is the point \( (0.5, 0.5) \) instead of \( (0, 0) \); i.e., its origin has been shifted by \( (0.5, 0.5) \). You should plot \( \mathbb{Z}^* \) to see what is going on here.

(2.3) Dual Percolation. Each edge \( e \) in \( \mathbb{Z}^2 \) intersects a unique edge in \( \mathbb{Z}^* \) halfway in the middle. We can call this latter edge the dual edge to \( e \). Whenever an edge in \( \mathbb{Z}^2 \) is open, its dual is declared close, and conversely, if an edge in \( \mathbb{Z}^2 \) is closed, we declare its dual edge
in $\tilde{Z}^2$ open. Clearly, this process creates a percolation process on the dual lattice $\tilde{Z}^2$, but the edge-probabilities are now $(1 - p)$ instead of $p$. Now if there is no percolation on $Z^2$, this means that on $\tilde{Z}^2$, there must exist an open “circuit” surrounding the origin. For a picture of this, see

\url{http://www.math.utah.edu/~davar/REU-2002/notes/lec5.html}

The probability that any given circuit, surrounding the origin, of length $n$ is dual-open is $(1 - p)^n$. So,

\begin{equation}
E \left[ \# \text{ of open circuits in } \tilde{Z}^2 \text{ of length } n \right] \leq C_n (1 - p)^n,
\end{equation}

where $C_n$ denotes the number of circuits—in $\tilde{Z}^2$—of length $n$ that surround the origin. Thus, we have shown that

\begin{equation}
P \{ \text{no percolation in } Z^2 \} \leq C_n (1 - p)^n.
\end{equation}

We want to show that is $p$ is large enough, the above goes to zero as $n \to \infty$. To do so, we need a bound for $C_n$.

\textbf{(2.6) Bounding $C_n$.} It is easier to count the number of circuits of length $n$ in $Z^2$ (not the dual) that surround the origin. This number is also $C_n$ (why?). But for a path $\pi := \pi_0, \ldots, \pi_n$ to be a circuit of length $n$ about $(0, 0)$, it must be that any $(n - 1)$ steps in $\pi$ form a self-avoiding path, and that $\pi$ must go through one of the points $(1, 0), (1, \pm 1), (1, \pm 2), \ldots, (1, \pm \lfloor \frac{n}{2} \rfloor)$. (There are at most $(n + 1)$ of these points.) Therefore, $C_n \leq (n + 1) \chi_{n-1}$ (why?) Recalling (1.3) above, and since $\chi_{n-1} \approx \{C(d)\}^{n-1}$, this and (2.5) show that whenever $p > 1 - \frac{1}{C(d)}$, then there can be no percolation, which is the desired result. \hfill \clubsuit