Solutions to midterm 1

1. (a) (2 marks) For \( f(x) = 3x - 1 \), write down \( f'(7) \).

**Solution:** \( f'(x) = 3 \) so \( f'(7) = 3 \).

(b) (4 marks) For \( y = \frac{3x^2 - 1}{\cos x} \), find \( \frac{dy}{dx} \).

**Solution:** The quotient rule gives

\[
\frac{dy}{dx} = \frac{\cos(x) \cdot (3x^2 - 1)' - (3x^2 - 1) \cdot (\cos(x))'}{\cos^2(x)}
\]

\[
= \frac{6x \cdot \cos(x) + (3x^2 - 1) \cdot \sin(x)}{\cos^2(x)}.
\]

This can be written in other ways but I'm happy with the above. If you simplify further than you may make more mistakes and so lose marks.

(c) (4 marks) For \( f(t) = (t^2 - 4t) \cdot \sqrt{t} \), find \( f'(t) \).

**Solution:** The product rule gives

\[
f'(t) = (t^2 - 4t) \cdot (\sqrt{t})' + (t^2 - 4t) \cdot \sqrt{t}' = \frac{t^2 - 4t}{2\sqrt{t}} + (2t - 4) \cdot \sqrt{t}.
\]

Again, this can be simplified further but I'm happy with this.

(d) (3 marks) Write down \( \int \cos x \, dx \).

**Solution:** An antiderivative \( F(x) \) is a function whose derivative equals \( \cos(x) \). But you already know one, namely \( F(x) = \sin(x) \). It follows that *all* antiderivatives are of the form \( \sin(x) + c \) for some constant \( c \), i.e.,

\[
\int \cos x \, dx = \sin x + c.
\]

(e) (7 marks) For \( f(x) = \sqrt{1 + \sin(2x + 3)} \), find \( f'(x) \).

**Solution:** The chain rule (applied twice) gives

\[
f'(x) = \frac{1}{2} (1 + \sin(2x + 3))^{-1/2} \cdot (1 + \sin(2x + 3))'
\]

\[
= \frac{1}{2} (1 + \sin(2x + 3))^{-1/2} \cdot \cos(2x + 3) \cdot (2x + 3)'
\]

\[
= \frac{1}{2} (1 + \sin(2x + 3))^{-1/2} \cdot \cos(2x + 3) \cdot 2
\]

Again this can be simplified (you'd be a little nuts not to notice that half times two is one which makes it look nicer) but I'm happy with the above.
2. (5 marks) Find the equation of the tangent line to the graph \( y = 5x^3 + 4 \) at the point \( x = 1 \). Write this line in the form \( y = mx + b \).

Solution: The derivative \( \frac{dy}{dx} \) is \( 15x^2 \), so the slope of the tangent line is 15 when \( x = 1 \). The \( y \)-value when \( x = 1 \) is \( y = 5 \cdot 1^3 + 4 = 9 \) so the line has equation
\[
y - 9 = 15(x - 1) \implies y = 15x - 6.
\]

3. (10 marks) Answer 'true' (T) or 'false' (F) by circling the appropriate letter.

T  “The derivative of \( \sec(x) \) is undefined at \( x = \pi/2 \).”

F  \[
\int_0^5 1 \, dt = 0.
\]

T  “If the graph of a function \( y = f(x) \) has a sharp corner at the point \((c, f(c))\) then the value \( f'(c) \) is undefined.”

T  “The derivative of \( \sin^2 x \) is \( 2\sin x \).”

?  “I think the webwork assignments are better than in-class quizzes (honest answer please, I’m genuinely interested).”

4. (7 marks) You’re driving through a forest on a dark night and your speedometer reads 160 feet per second. Fee-fee, a cute french poodle, walks onto the road 801 feet ahead. Slamming on your brakes causes you to decelerate at 16 feet per second per second.

(a) (5 marks) Do you hit Fee-fee? Please explain.

Solution: First set acceleration \( a(t) = -16 \). Then \( v(t) = -16t + c \) for some constant \( c \). If we decide to start the clock from the moment we see Fee-fee it follows that \( v(0) = 160 \). Substitute \( t = 0 \) into our equation for \( v(t) \) gives \( c = 160 \), i.e.,
\[
v(t) = -16t + 160.
\]

Now repeat. By integrating \( v(t) \) we get \( s(t) = -8t^2 + 160t + c \), where \( c \) is some new constant. To work out what \( c \) is, we can decide that we start measuring distance \( s(t) \) from the point when we first spot Fee-fee, i.e., \( s(0) = 0 \). The we have that \( c = s(0) = 0 \) so that
\[
s(t) = -8t^2 + 160t.
\]

The car stops when \( v(t) = 0 \), i.e., when \(-16t + 160 = 0\), hence when \( t = 10 \) seconds. Substitute this into our equation for \( s(t) \) to conclude that the car stops after going exactly \( s(10) = -8(10)^2 + 160(10) = 800 \) feet. But initially we were 801 feet away from Fee-fee and since we travel only 800 feet we (just!) miss the terrified poodle.

\[1\text{If you don’t like using the same letter twice in the same question, feel free to write } d \text{ in the formula for } s(t) \text{ in place of the letter } c. \text{ Some lecturers might be pedantic about this, I choose not to be.} \]
(b) (2 marks) If you think that the car hits Fee-fee (would I be so cruel?), write down the velocity of the car the moment that it strikes Fee-fee. If you think that the car doesn’t hit Fee-fee, how far from your car bumper is the terrified pooch when you finally come to a halt?

**Solution:** We stop 1 foot from Fee-fee. If you thought that we struck Fee-fee then you’ll get the marks if you do the velocity calculation correctly.

5. (8 marks)

(a) (2 marks) Write down the limit definition of the derivative \( f'(x) \) of a function \( f(x) \):

**Solution:** \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \). There are other forms that are correct and I’ll accept any of these.

(b) (6 marks) Find the derivative of \( f(x) = \frac{1}{x + 6} \) using the limit definition.

**Solution:**

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{x + h + 6} - \frac{1}{x + 6} \right) = \lim_{h \to 0} \frac{1}{h} \left( \frac{x + 6 - (x + h + 6)}{(x + 6)(x + h + 6)} \right) = \lim_{h \to 0} \frac{1}{h} \frac{-h}{(x + 6)(x + h + 6)} = \lim_{h \to 0} \frac{-1}{(x + 6)(x + h + 6)} = -\frac{1}{(x + 6)^2}
\]

so the final answer is \( f'(x) = -1/(x + 6)^2 \), which you can compute directly using the chain rule.