Graph Drawing [§4.2, 4.3, 4.6]

I propose the following six step guide for drawing the graph $y = f(x)$ of a rational function. In fact the same method will work for almost any function with minor modifications.

1. Analysis involving $f$:
   
   (a) Find all zeros and vertical asymptotes, i.e., points where $f$ may change from positive to negative, or vice versa. These are points where $f(x) = 0$ and where $f(x)$ undefined.

2. Analysis involving $f'$:
   
   (a) Compute $f'(x)$.

   (b) Find all critical points, i.e., the points where $f$ may change from increasing to decreasing, or vice versa. That is, points where $f'(x) = 0$ or where $f'(x)$ undefined.

   (c) Draw a sign diagram for $f'(x)$ to learn where $f$ is increasing and decreasing.

3. Analysis involving $f''$:
   
   (a) Compute $f''(x)$.

   (b) Find all potential inflection points, i.e., the points where $f$ may change from concave up to concave down, or vice versa. That is, points where $f''(x) = 0$ or where $f''(x)$ undefined.

   (c) Draw a sign diagram for $f''(x)$ to learn where $f$ is concave up and concave down.

4. Work out the behaviour of $f(x)$ as $x \to \pm \infty$ (see §2.8).

5. Plot the points/asymptotes obtained in steps 1,2 and 3 above.

6. Between each point/asymptote you know that $f$ is increasing/decreasing and concave up/concave down so there is only one way to join the dots!

Another interesting topic is the following question:

"Given the graph of $y = f'(x)$ what does the graph of $y = f(x)$ look like?".

I recommend that you draw sign diagrams for $f'(x)$ and $f''(x)$ directly underneath the graph of $y = f'(x)$. Now draw $y = f(x)$ as above.