Answers to Midterm 1

1. [10 points ] Two simple questions to begin:

   (a) Find the slope of the straight line joining the points (3,6) and (5,1), then write down the equation of this straight line in the form \( y = mx + b \).

   **Solution:** The slope is \( m = (1 - 6)/(5 - 3) = -5/2 \). Sub into the point-slope formula to get
   \[
   y - 1 = -\frac{5}{2} \cdot (x - 5) \implies y = -\frac{5}{2}x + \frac{27}{2}.
   \]

   (b) Completely *factorise* the expression \( x^3 + 3x^2 + x + 3 \), then write down the zeros (both real and complex) of the function \( f(x) = x^3 + 3x^2 + x + 3 \).

   **Solution:** Notice the factor \( x + 3 \) so
   \[
   x^3 + 3x^2 + x + 3 = (x + 3)(x^2 + 1) = (x + 3)(x - i)(x + i).
   \]
   The zeros of \( f(x) \) are therefore \( x = -3, \pm i \).

2. [10 points ] Consider \( f(x) = (x - 2)^3 + 1 \) and \( g(x) = (x - 2)^3 + 1 \).

   (a) Draw the graphs \( y = f(x) \) and \( y = g(x) \). Indicate the \( y \)-intercept and at least one other point on each graph to convince me that you’ve got the idea!

   **Solution:** Begin with the graph \( y = x^2 \), shift to the right 2, then shift up 1. I would mark the vertex at (2,1) and the \( y \)-intercept (0,5). The case for \( g \) is similar, but of course you begin with \( y = x^2 \) and the \( y \) intercept of the graph of \( g \) is (0,-7).

   (b) Write down the domain and range of both \( f \) and \( g \).

   **Solution:** Domain \( f = \mathbb{R} \), range \( f = \{ y \in \mathbb{R} \mid y \geq 1 \} \), domain \( g = \mathbb{R} \), range \( g = \mathbb{R} \).

   (c) Precisely one of \( f \) and \( g \) has an inverse. Which one? Find the inverse of this function algebraically.

   **Solution:** \( g \) has an inverse. To find this, solve for \( x \) as a function of \( y \):
   \[
   y = (x - 2)^3 + 1 \implies y - 1 = (x - 2)^3 \implies \sqrt[3]{y - 1} = x - 2
   \]
   so the inverse of \( g \) is the function \( g^{-1}(y) = \sqrt[3]{y - 1} + 2 \).

3. [10 points ] Circle T or F according to whether you think the statement is True or False:

   \[ \text{T} \quad y = 6x^2 + \frac{1}{2} \text{ has no } x \text{-intercepts.} \]

   **Solution:** It’s the graph of \( y = x^2 \) shifted up, so no \( x \)-intercepts.
F \quad (7x + 4) \text{ is a factor of a function } f(x) \text{ if and only if } \frac{1}{7} \text{ is a zero of } f(x).

\textbf{Solution:} (7x + 4) \text{ is a factor if and only if } -\frac{1}{4} \text{ is a zero}

T \quad (2x - 1) \text{ is a factor of } 6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48.

\textbf{Solution:} Just do the synthetic division to check.

F \quad \text{No complex number is equal to its conjugate.}

\textbf{Solution:} Any real number } a \in \mathbb{R} \subset \mathbb{C} \text{ is equal to its complex conjugate.}

F \quad \text{Every straight line is the graph of a function.}

\textbf{Solution:} Any vertical line is defined by an equation } x = a \text{ for some } a, \text{ but this is not the graph of a function.}

4. [10 points]

(a) Use long division (any synthetic division junkies will be flogged!) to show that } x = \sqrt{2} \text{ is a solution of the polynomial equation}

\[ x^3 + 2x^2 - 2x - 4 = 0. \]

\textbf{Solution:} Divide } x^3 + 2x^2 - 2x - 4 \text{ by } x - \sqrt{2}. \text{ The synthetic division is}

\[
\begin{array}{c|cccc}
\sqrt{2} & 1 & 2 & -2 & -4 \\
0 & \sqrt{2} & 2\sqrt{2} + 2 & 4 \\
1 & 2 + \sqrt{2} & 2\sqrt{2} & 0 \\
\end{array}
\]

so } x = \sqrt{2} \text{ is a zero of the polynomial } x^3 + 2x^2 - 2x - 4 \text{ or, to put it another way, } x = \sqrt{2} \text{ is a solution of the equation } x^3 + 2x^2 - 2x - 4 = 0.

(b) Use your result to part (a) to completely factor } x^3 + 2x^2 - 2x - 4.

\textbf{Solution:} Part (a) shows that } f(x) = (x - \sqrt{2})(x^2 + (2 + \sqrt{2})x + 2\sqrt{2}). \text{ Now factor the quadratic part to leave}

\[ f(x) = (x - \sqrt{2})(x + 2)(x + \sqrt{2}). \]

(c) Determine the right-hand and left-hand behaviour of the graph

\[ y = x^3 + 2x^2 - 2x - 4, \]

i.e., state whether the graph rises or falls as you go far to the left and to the right.

\textbf{Solution:} It behaves like } y = x^3 \text{ far to the right and left. This rises far to the right and falls far to the left.
5. [10 points] Find the zeros of the function \( f(x) = x^4 + 6x^3 + 10x^2 + 6x + 9 \), and write \( f(x) \) as a product of linear factors.

**Solution:** There’s no obvious way to factor, so use the rational zero test. It tells you that possible rational zeros are \( \pm 1, \pm 3, \pm 9 \). Substitute these numbers in to see that \( f(-3) = 0 \). Now use synthetic division:

\[
\begin{array}{c|cccc}
-3 & 1 & 6 & 10 & 6 & 9 \\
& & 0 & -3 & -9 & 9 \\
\hline
& 1 & 3 & 1 & 3 & 0
\end{array}
\]

so

\( f(x) = (x + 3)(x^3 + 3x^2 + x + 3) \).

My hope was that you would notice that \( x^3 + 3x^2 + x + 3 \) is the polynomial from question 1(b) so you’ve already found it’s factorisation! So

\[
f(x) = (x + 3)(x + 3)(x - i)(x + i).
\]

The zeros of \( f \) are therefore \( x = 3, \pm i \).

6. [10 points] A Precalculus Solutions Manual is projected directly upwards from the ground with an initial velocity of 128 feet per second. Its height \( h \) at time \( t \) is approximately \( h(t) = 128t - 16t^2 \).

(a) Find the maximum height of the evil book by completing the square.

**Solution:**

\[
h(t) = -16t^2 + 128t \\
= -16(t^2 - 8t) \\
= -16(t^2 - 8t + 16 - 16) \\
= -16(t - 4)^2 - 16 \cdot (-16) \\
= -16(t - 4)^2 + 256
\]

Maximum point occurs at the vertex \( (4, 256) \), so max height is 256.

(b) When does it hit the ground?

**Solution:** Ground is height zero, so solve \( h(t) = 0 \), i.e.,

\[
0 = 128t - 16t^2 = 16t \cdot (8 - t).
\]

Either \( t = 0 \) or \( t = 8 \). Of course the first solution is the moment that the manual is projected up, so the second solution \( t = 8 \) is when it hits the ground.

(c) Using your answers to parts (a) and (b), draw the graph of the function \( h(t) = 128t - 16t^2 \), clearly indicating the \( y \)-intercept, the \( x \)-intercept(s) and the vertex.

**Solution:** The \( y \)-intercept is \( (0,0) \), the \( x \)-intercepts are \( (0,0), (8,0) \) and the vertex is \( (4, 256) \) so join the dots with a smooth parabola.