Basic Geometric Representation Theory

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1 Introduction

Dating back to the early 1900s, representation theorists have tried to classify all of the irreducible representations of semisimple and reductive Lie groups (e.g. $SL_n(\mathbb{R})$ or $GL_n(\mathbb{R})$). This problem is unwieldy and in some sense larger than what the original motivation dictated should be studied. In fact, Gel'fand in the 1920s determined that we should focus on classifying *unitary* representations. It was then Harish-Chandra's incite that this class of representations is contained in a larger class called *admissible* (or quasi-simple via a difficult theorem of Harish-Chandra). In particular, due to his work we obtained inclusions

$$\widehat{G}_{temp} \subseteq \widehat{G}_{unit} \subseteq \widehat{G}_{adm}$$

where the first inclusion will be made more precise later. In showing these, Harish-Chandra also introduced the notion of infinitesimal representations arising from the group representation. These mirrored the techniques in the finite dimensional case of passing to the Lie algebra. Further, he showed that we have the following fact:

Theorem 1.1 (Harish-Chandra)**.** *Let (π, V) be an admissible representation of G. Then V is irreducible if and only if it is infinitesimally irreducible.*

Thus, studying irreducible admissible representations is the same as studying irreducible admissible infinitesimal representations. The latter is partially classified by a theorem of Langlands:

Theorem 1.2 (Langlands)**.** *Let (π, V) be an irreducible admissible infinitesimal representation of G. Then there exists a parabolic subgroup $P = MAN$ of G, a tempered representation ω of M, and a character, ν, of A whose real part lies in the open anti-dominant Weyl chamber such that*

$$V \hookrightarrow \mathrm{Ind}_P^G(\omega \otimes \nu \otimes 1)_K$$

(where $(-)_K$ means passing to the infinitesimal representation).

Langlands' theorem reduces the study of admissible representations to the study of tempered ones. It was shown by Vogan that there is a bijection for tempered representations with real infinitesimal character:

Theorem 1.3 (Vogan)**.** *Let $\widehat{G}_{temp,\mathbb{R}}$ denote the isomorphism classes of irreducible tempered representations with real infinitesimal character. Then*

$$\widehat{G}_{temp,\mathbb{R}} \xrightarrow{\sim} \widehat{K}$$

where K is a maximal compact subgroup of G (e.g. $K = O(n)$ for $G = GL(n, \mathbb{R})$).

This is the beginning of the geometric theory in some sense. $\widehat{K}$ is completely classified by a theorem of Borel and Weil:

Theorem 1.4 (Borel-Weil)**.** *Let K be a compact Lie group and $T \subseteq K$ be a maximal torus. Let $H = K_{\mathbb{C}}$ be the complexification and $B = MA\overline{N}$ a Borel subgroup. Then the irreducible finite dimensional representations of K stand in one-to-one correspondence with the dominant, analytically integral weights $\lambda \in \mathfrak{t}^*$ with the correspondence given by*

$$\lambda \mapsto \Gamma_{Hol}(K/T, L_\lambda) \cong \mathscr{F}^{Hol}_{B,\chi_\lambda}$$

where $\Gamma_{Hol}(K/T, L_\lambda)$ denotes the set of holomorphic sections of the bundle and

$$\mathscr{F}^{Hol}_{B,\chi_\lambda} = \left\{ f : G \to \mathbb{C} \,\middle|\, f(gb) = \chi_\lambda(b)^{-1} f(g), f \text{ holomorphic} \right\}$$

with χ_λ the character of B associated to the analytically integral weight λ.

In particular, we see that we get a bijection $\widehat{K} = \Lambda_+(T)$ the set of dominant integral weights of T. Equivalently we could phrase this bijection in terms of certain K-equivariant holomorphic vector bundles on the flag variety K/T. What we actually seek is a geometric classification of representations of K in terms of representations of G. To have any chance of this existing, we need to first determine how to construct geometric parameters for such a classification.

2 Irreducible Representations of G and their $(\mathfrak{g}, K)$-modules

To make perfect sense of what will follow, we will need some general facts about the infinitesimal representation s referenced in the introduction.

We now return to the general case. What we present here again will be true for groups in the Harish-Chandra class, but the proofs will be presented for reductive groups unless noted otherwise. Let G be a real reductive Lie group, $\mathfrak{g}_0$ its Lie algebra, and $\mathfrak{g}$ its complexification. Let Θ be a global Cartan involution and K the maximal compact subgroup corresponding to the choice of Θ. We have the following decompositions of G:

$$\begin{aligned} G &= K \times \mathfrak{p}_0 && \text{Polar Decomposition} \\ G &= KAN && \text{Iwasawa Decomposition} \\ G &= KAK && \text{Cartan Decomposition} \end{aligned}$$

Let (Π, V) be any infinite dimensional complex representation of G on a Hilbert (or even Banach, Fréchet, etc.) space V with inner product $\langle , \rangle$. Unlike the finite dimensional case, the following limit may not exist:

$$\lim_{t \to 0} \frac{\Pi(\exp tX)v - v}{t} \tag{1}$$

for $g \in G$ and $v \in V$.

Definition 2.1. We say that a vector $v \in V$ is of **class** C^1 if the mapping $v \mapsto \lim_{t\to 0} \frac{\Pi(\exp tX)v - v}{t}$ is continuous. We call a vector **smooth** or of class C^∞ if the mapping

$$v \mapsto \Pi(g)v$$

is C^k for all $k \geq 0$. Let $V^\infty \subseteq V$ denote the subspace of all smooth vectors.

Lemma 2.2. (Π, V^∞) *is a representation of G. Further, we can differentiate this action to get a representation (π, V^∞) of $\mathfrak{g}$.*

Proof. Let $v \in V^\infty$. Then for all $g, h \in G$, $gh \mapsto \Pi(gh)v$ is C^∞. As Π is a homomorphism, $\Pi(g)\Pi(h)v \in V^\infty$ and by associativity

$$\Pi(gh)v = \Pi(g) \cdot (\Pi(h)v) \in V^\infty$$

Thus, $\Pi(h)v \in V^\infty$. As $g, h \in G$ are arbitrary, we see that $\Pi(g)v \in V^\infty$ for all $v \in V^\infty$ and (Π, V^∞) is a (smooth) representation of G.

To define the Lie algebra representation, let $X \in \mathfrak{g}_0$ and for each $v \in V^\infty$ put

$$\pi(X)v = \lim_{t \to 0} \frac{\Pi(\exp tX)v - v}{t} = \frac{d}{dt}\Pi(\exp tX)v|_{t=0}$$

This is well-defined as $v \in V^\infty$ and also leaves V^∞ invariant as $\Pi(\exp tX)v$ is smooth. It remains to show that π is a Lie algebra homomorphism. (Note that if V is finite dimensional then every vector is smooth and π is a Lie algebra homomorphism as it is the tangent map to a Lie group homomorphism)

Let $X \in \mathfrak{g}_0$ and put $\tilde{X}$ the corresponding left-invariant vector field on G. Then if we put $f_v(g) = \Pi(g)v$, we see that

$$\Pi(g)\pi(X)v = (\tilde{X}f_v)(g)$$

By putting $g = \exp tY$ for $Y \in \mathfrak{g}_0$, we obtain

$$\pi(Y)\pi(X)v = \frac{d}{dt}\Big|_{t=0}\Pi(\exp tY)\pi(X)v = \frac{d}{dt}\Big|_{t=0}(\tilde{X}f_v)(\exp tY) = \tilde{Y}(\tilde{X}f_v)(e)$$

Interchanging Y and X and subtracting them, we get

$$\pi(X)\pi(Y)v - \pi(Y)\pi(X)v = (\tilde{X}(\tilde{Y}f_v) - \tilde{Y}(\tilde{X}f_v))(e)$$

The right hand side is precisely $([\tilde{X}, \tilde{Y}]f_v)(e) = \widetilde{[X,Y]}f_v(e)$ and thus π is a Lie algebra representation. As V^∞ is a complex vector space, we see that π extends to a complex Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(V^\infty)$ and thus an associative algebra homomorphism $U(\mathfrak{g}) \to \mathrm{End}(V^\infty)$ which sends 1 to 1. ☐

One key aspect of this procedure is that V^∞ is not some arbitrary subspace of V as the following example shows:

Example 2.3. Let $G = \mathbb{R}$ and $(\lambda, L^2(\mathbb{R}))$ be the left-regular representation of G. Put $V = \{f \in C_c^\infty(\mathbb{R}) : \operatorname{supp} f \subseteq [0,1]\}$. Then V is $\mathfrak{g}$ invariant, but neither V, nor its closure are G invariant.

What we want to show is that V^∞ is not only a suitably nice subspace, but that $V = \overline{V^\infty}$. To do this, first will extend our given representation Π from G to $C_c^\infty(G)$. From here, it will then follow that the set of all matrix coefficients of this extension will be dense in V and will all be smooth. We start with the extension.

Definition 2.4. Let (Π, V) be a Hilbert space representation of G. For any $f \in C_c^\infty(G)$, put

$$\Pi(f)v = \int_G f(g)\Pi(g)v \, dg$$

where dg denotes a left-Haar measure on G. Notice that $||\Pi(f)v|| \le C_\Omega ||v|| ||f||_1$ for all compact $\Omega \subseteq G$, where C_Ω is a constant. Define the **Gårding Subspace** of V to be the linear span of the $\Pi(f)v$.

The reason we consider such a space is that we can chose a sequence of f_i to be an *approximation to the identity* (a net of distributions which converges to the identity element). This in turn tells us that at least some of the $\Pi(f)v$ are smooth vectors. As the following proposition shows, in fact all of the $\Pi(f)v$ are smooth. Even more convenient, the G̊ding subspace is dense in V and by extension V^∞ is dense.

Proposition 2.5. *Let (Π, V) be a representation of G and $f \in C_c^\infty(G)$. Then the following are true:*

(a) For every $v \in V$, $\Pi(f)v \in V^\infty$.

(b) The Gårding subspace of V is dense in V.

(c) V^∞ is dense in V.

Proof. For (a), we first show that for $X \in \mathfrak{g}$ then $\pi(X)\Pi(f)v$ exists. By the definitions:

$$\pi(X)\Pi(f)v = t^{-1} \int_G f(g) \left(\Pi(\exp tX)\Pi(g) - \Pi(g)\right) \, dg$$

By making the substitution $g \mapsto \exp(-tX)g$, we get that

$$t^{-1} \int_G f(g) \left(\Pi(\exp tX)\Pi(g) - \Pi(g)\right) \, dg = \int_G \frac{f(\exp(-tX)g) - f(g)}{t} \Pi(g)v \, dg$$

By taking the limit as $t \to 0$ and applying the Dominated Convergence Theorem, we see that $\pi(X)\Pi(f)v = -\Pi(Xf)v$ and thus exists. This shows that the Gårding subspace is stable under $\mathfrak{g}$ and thus consists of smooth vectors.

For (b), let $v \in V$. By assumption Π is continuous and thus for any $\varepsilon > 0$ the set

$$T = \{g \in G : ||\Pi(g)v - v|| < \varepsilon\}$$

is open. Therefore, there exists some $C \subseteq T$ compact and $f \in C_c^\infty(G)$ with $\operatorname{supp} f \subseteq C$. By normalizing, we can assume $\int_G f \, dg = 1$ and thus

$$||\Pi(f)v - v|| = || \int_G f(g)[\Pi(g)v - v] \, dg|| \leq \int_G f(g)||\Pi(g)v - v|| \, dg \leq \varepsilon \int_G f(g) \, dg = \varepsilon$$

Hence, the Gårding subspace is dense in V. Part (c) follows from (a) and (b). This completes the proof. $\square$

We now want to mirror the finite-dimensional case. The main obstruction to directly doing this being that characters no longer entirely determine the representation (namely because they do not exist in the traditonal sense) and thus only the $U(\mathfrak{g})$-module structure on V^∞ is not sufficient to reconstruct the G-module structure on V. This is mainly a topological obstruction. Notice however, that we have a small work-around for this by using the Polar (Cartan) decomposition of G. It shows that $K \simeq G$ and thus morally the representation theory of G should come from the representation theory of K, possibly with some twist. As it will turn out, this yoga is true for irreducible representations! As understanding the irreducible representations of G are the real goal of representation theory, this is exactly the result we want.

The "twist" mentioned above is that we shall consider a certain subspace of V which is generated by suitably nice vectors which allow us to decompose V as an algebraic direct sum of irreducible spaces. This will in turn be the correct space to study.

Definition 2.6. A vector $v \in V$ is called K-**finite** if $\Pi(K)v$ spans a finite dimensional vector space. The subspace of all K-finite vectors is denoted V_K.

If (Π, V) is a representation of G, then we can investigate $\Pi|_K$. If K acts by unitary operators, then $(\Pi|_K, V)$ decomposes as a Hilbert space direct sum of irreducible representations of K. For each $\gamma \in \widehat{K}$, denote by $V(\gamma)$ the sum of all of the subrepresentations isomorphic to γ. This is called the $\gamma -$ **isotypic component** of V.

Definition 2.7. A representation of G is called **admissible** if $\dim_{\mathbb{C}} V(\gamma) < \infty$ for all $\gamma \in \widehat{K}$. Notice that this is equivalent to $m_\gamma(\Pi|_K) < \infty$.

Proposition 2.8. *Let (Π, V) be an admissible representation of G. Then $V_K \subseteq V^\infty$.*

Proof. We shall proceed in a few steps.

Step 1: Notice that $V_K \cap V^\infty \neq \varnothing$ as V^∞ is dense in V. We shall momentarily call vectors in this intersection: smooth K-finite vectors for V. Let us show that a special class of functions give K-finite vectors. Consider λ the left-regular representation of K on $C^\infty(K)$. Denote by λ_{fin} the K-finite vectors in $C^\infty(K)$ and let $f \in \lambda_{\text{fin}}$. Further, let $h \in C_c^\infty(\exp \mathfrak{p}_0)$ and put

$$F(k \exp \mathfrak{p}_0) = f(k)h(\exp X)$$

Each F of this form is compactly supported and *left* K-finite for λ as

$$\lambda(k_0)F(k \exp X) = F(k_0^{-1}k \exp X) = f(k_0^{-1}k)h(\exp X)$$

where the right hand side is K-finite by assumption.

Now, for every $v \in V$, $\Pi(F)v \in V^\infty$ by the proof of Proposition 2.5. Unraveling the definitions a bit, we see that

$$\Pi(k_0)\Pi(F)v = \int_G F(g)\Pi(k_0 g)v \, dg = \int_G F(k_0^{-1}g)\Pi(g)v \, dg = \lambda(k_0^{-1})\Pi(F)v$$

with the right hand side K-finite by above. Hence, $\Pi(F)v$ is K-finite.

Step 2: Now we shall show that the linear span of all $\Pi(F)v$ is dense in V. Let $v \in V$ be arbitrary. As Π is continuous, the set

$$T = \{g \in G : ||\Pi(g)v - v|| < \varepsilon\}$$

is open and contains some compact subset C. We may then choose f, h supported in C such that $\int_G F(g) \, dg = 1$. Then

$$\begin{aligned} ||\Pi(F)v - v|| &= \left|\left| \int_G F(g)\Pi(g)v \, dg - v \right|\right| \\ &\leq \int_G F(g) \; ||\Pi(g)v - v|| \, dg \\ &< \varepsilon \int_G F(g) \, dg = \varepsilon \end{aligned}$$

Hence, the linear span of all such $\Pi(F)v$ is dense in V. This shows that the set of smooth K-finite vectors (namely $V_k \cap V^\infty$) is dense in V.

Step 3: Now we show that for admissible representations this subspace this is precisely all of the K-finite vectors. As V is assumed admissible, write $\Pi_K = \widehat{\bigoplus}_{\gamma \in \widehat{K}} V(\gamma)$ with each

$\dim_{\mathbb{C}} V(\gamma) < \infty$. Consider $V(\gamma)$. As this is finite dimensional and irreducible, it consists entirely of K-finite vectors. As this is true for every such γ, we see that $V_K = \bigoplus_{\gamma \in \widehat{K}} V(\gamma)$ (the algebraic direct sum). Combining this with the result from above, we have a dense subspace of a linear space which in turn must be the entire space. Since, the K-finite vectors are finite linear combinations of elements of the $V(\gamma)$, we see that $V_K = V_K \cap V^\infty$. Hence, all K-finite vectors are smooth. □

Similar to the smooth case above. Harish-Chandra proved that all K-finite vectors are in fact real analytic by way of matrix coefficients and elliptic differential operators. The main tool in that approach is the following fact which we shall not prove:

Theorem 2.9. *Let D be an elliptic differential operator with real-analytic coefficients. Then every element in the solution space $Df = 0$ is real-analytic.*

It can be shown that K-finite matrix coefficients satisfy a certain elliptic differential operator and thus are real-analytic. For a proof of this, see [Kna86]

Lemma 2.10. *V_K is a $\mathfrak{g}$ invariant subspace of V^∞.*

Proof. Let $v \in V_K$ and put $W_v = \pi(U(\mathfrak{k}_{\mathbb{C}}))v$ the necessarily finite dimensional subspace corresponding to v. Then for $X \in \mathfrak{k}, Y \in \mathfrak{g}$, and $v' \in W_v$, we have that

$$\pi(X)\pi(Y)v' = \pi(Y)\pi(X)v' - \pi([X,Y])v'$$

and $\pi(\mathfrak{g})v'$ is stable under $\pi(\mathfrak{k})$. As W_v is finite dimensional, we can exponentiate the elements and conclude that $\pi(\mathfrak{g})W_v$ is $\Pi(K)$-invariant. Thus, $Y \in \mathfrak{g}$ implies that $\pi(Y)v$ is an element of a finite dimensional vector space which is $\Pi(K)$-stable. Hence, $\pi(Y)v$ is K-finite □

Therefore V_K carries two related representations: one of $\mathfrak{g}$ and one of K. As it turns out, vector spaces with this property are incredibly rich in structure and thus are the next object of study. We formalize this in the following definition.

Definition 2.11. (Lepowsky) Let G be a non-compact reductive Lie group, $G = KAN$ the Iwasawa decomposition. A $(\mathfrak{g}, K)$-**module** is a vector space V equipped with two representations, denoted by $\pi : \mathfrak{g} \cup K \to \mathrm{End}(V)$, of $\mathfrak{g}$ and K, such that the following conditions are satisfied:

(a) Every $v \in V$ is K-finite.

(b) The differential of $\pi|_K$ is the restriction of $\pi_{\mathfrak{g}}|_{\mathfrak{k}_0}$.

(c) For all $k \in K$ and $X \in \mathfrak{g}$, we have that $\pi(\mathrm{Ad}(k)X) = \pi(k)\pi(X)\pi(k)^{-1}$.

If V is equipped with an inner product, we say that V is a **unitary** $(\mathfrak{g}, K)$-module if $\pi(X)$ is a unitary operator for all $X \in \mathfrak{g}$. We say that V is an **admissible** $(\mathfrak{g}, K)$-module if for every $\gamma \in \widehat{K}$, the γ-isotypic component $V(\gamma)$ is finite dimensional.

Corollary 2.12. *If (Π, G) is an admissible representation of G, then (π, V_K) is a $(\mathfrak{g}, K)$-module by Lemma 2.10.*

2.1 $(\mathfrak{g}, K)$-modules

This next section will cover the basics of the $(\mathfrak{g}, K)$-module theory. In particular, we shall see that irreducible $(\mathfrak{g}, K)$-modules completely determine the irreducible G-modules. This is the correspondence hinted at in the introduction. We have not proven this directly, but it can be shown that the assignment $V \mapsto V_K$ is functorial and this functor is fully faithful on the category of irreducible $(\mathfrak{g}, K)$-modules (up to equivalence).

2.1.1 Admissible and Unitary representations

We have mirrored the finite dimensional theory fairly closely to this junction. One result which has been starkly absent is Schur's Lemma. The main reason for this is that in the infinite dimensional setting, it may not hold! Luckily, if we restrict ourselves to *countably* infinite bases, we get an analogous result due to Dixmier.

Lemma 2.13. *Let V be a countable dimensional $\mathbb{C}$-vector space and suppose $T \in \operatorname{End}(V)$. Then there exists a scalar $c \in \mathbb{C}$ such that $T - cI$ is not invertible.*

Proof. Suppose for the sake of contradiction that $T - cI$ is invertible for all $c \in \mathbb{C}$. Then for every polynomial P, $P(T)$ is invertible as a function of one variable. Now, let $R(T) = P(T)/Q(T)$ be some rational function. This gives a linear map $\mathbb{C}(x) \to \operatorname{End}(V)$. For all $v \in V$, we have that $R(T)v = 0$ only if $P(T)v = 0$. Thus the map $\mathbb{C}(x) \to V$ given by $R \mapsto R(T)v$ is injective. Since $\mathbb{C}(x)$ is uncountably infinite dimensional over $\mathbb{C}$, we have a contradiction. ☐

Lemma 2.14. *(Dixmier) Suppose $S \subseteq \operatorname{End}(V)$ is a subset of endomorphisms which acts irreducibly. If $T \in \operatorname{End}(V)$ commutes with all elements of S, then $T = cI$ for some $c \in \mathbb{C}$.*

Proof. By the preceding lemma, there exists some $c \in \mathbb{C}$ such that $T - cI$ is not invertible. Consider $\ker(T - cI)$ and $\operatorname{Im}(T - cI)$. Every element of S preserves both of these spaces one of them is necessarily a proper subset. As $T - cI$ is not invertible, $\ker(T - cI)$ is nonzero. Thus $\ker(T - cI) = V$ as all of the elements of S act irreducibly. Hence $T - cI = 0$ and $T = cI$. ☐

Definition 2.15. If V and W are two $(\mathfrak{g}, K)$-modules, denote by $\operatorname{Hom}_{\mathfrak{g},K}(V, W)$ the set of all $\mathfrak{g}$ homomorphisms $V \to W$ which are also K homomorphisms.

Lemma 2.16. *Let V be an irreducible $(\mathfrak{g}, K)$-module. Then $\operatorname{Hom}_{\mathfrak{g},K}(V, V) = \mathbb{C} \cdot \operatorname{Id}$.*

Proof. Let $v \in V$ and W_v the span of v under K. Then $U(\mathfrak{g})W_v$ is a $\mathfrak{g}$-invariant, K-invariant subspace and thus $V = U(\mathfrak{g})W_v$. This exhibits V as a countably infinite dimensional space. By applying the above lemmas, we are done. ☐

The main motivation for studying representations began with Harmonic analysis where unitary representations arise naturally. In fact, all of the theory developed so far was originally done precisely to understand the *Unitary dual* of reductive Lie groups. Based on the discussion above, one may guess that admissible representations are related to unitary representations in some way. In fact, the precise statement is as follows:

Theorem 2.17. *Let G be a real reductive Lie group and (π, V) an irreducible unitary representation of G. Then (π, V) is admissible as a representation of G and thus V_K is an admissible, unitary $(\mathfrak{g}, K)$-module.*

Before getting to the proof, we need a few technical results, some of which we shall not prove for brevity. All of the statements can be found in [HC53]. Let $U(\mathfrak{k})$ denote the enveloping algebra of $(\mathfrak{k}_0)_{\mathbb{C}}$ considered as a subalgebra of $U(\mathfrak{g})$ and $\mathfrak{Y}$ a left ideal in $U(\mathfrak{k})$ such that $U(\mathfrak{k})/\mathfrak{Y}$ is finite dimensional and $\mathfrak{k}$ acts by semisimple transformations. Consider $X = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{Y}$. $U(\mathfrak{g})$ acts on this space by left-translation and thus restricts to a representation of $Z(\mathfrak{g})$ (the center of $U(\mathfrak{g})$. Additionally, using the adjoint representation, we get an action of K on this quotient. Let $X(\gamma)$ denote the γ-isotypic component of X.

Lemma 2.18. *For every $\gamma \in \widehat{K}$, $X(\gamma)$ is a finite module over $Z(\mathfrak{g})$.*

Proof. See [HC53, Theorem 1]. □

Lemma 2.19. *Let (π, V) be an admissible representation of G and V_K the associated $(\mathfrak{g}, K)$-module with $\psi \in V_K$. Then the closure $\overline{\pi(U(\mathfrak{g})\psi)}$ is $\pi(G)$ invariant.*

Proof. Let $\psi_0 \in U = \pi(U(\mathfrak{g})\psi$, and $\lambda \in V^*$ such that λ vanishes on U. Since $\psi_0 \in V_K$, we see that the map

$$g \mapsto \lambda(\pi(g)\psi_0)$$

is real analytic on G. Now, there exists a neighbourhood O of 0 in $\mathfrak{g}_0$ such that the exponential map is given by a power series and thus

$$\pi(\exp X)\psi_0 = \sum_{m=0}^{\infty} \frac{1}{m!}\pi(X)^m\psi_0$$

and thus

$$\lambda(\pi(\exp X))\psi_0) = \sum_{m=0}^{\infty} \frac{1}{m!}\lambda(\pi(X)^m\psi_0)$$

As $\pi(X)^m\psi_0 \in U$, the right hand side is 0. Therefore, $\lambda(\pi(g)\psi_0)$ vanishes on a neighbourhood of 1 in G and by analyticity, vanishes everywhere. Applying the Hahn-Banach theorem, we see that $\pi(g)\psi_0 \in \overline{U}$. Therefore $\pi(g)U \subseteq \overline{U}$ and by continuity, $\pi(g)\overline{U} \subseteq \overline{U}$. This completes the proof. □

Lemma 2.20. *Let (π, V) be a representation of G which admits an infinitesimal character. Then for any K-invariant subspace W, put $W(\gamma) = W \cap V(\gamma)$. If $\bigoplus_{\gamma\in\widehat{K}} W(\gamma)$ is dense in V, then $V(\gamma) = \overline{W(\gamma)}$.*

Proof. See [HC53, Lemma 30]. □

Remark 2.21. For a moral proof of the above lemma, mimic the ideas of the proof that the Gårding subspace is dense. Consider the integral operators associated to certain smooth functions and use the density of these to conclude the desired lemma.

Proposition 2.22. *Let (π, V) be a representation of G on a Hilbert space which admits an infinitesimal character. Let $\psi_0 \in V_K = \bigoplus_{\gamma\in\widehat{K}} V(\gamma)$ and $U = \overline{\pi(U(\mathfrak{g})\psi_0)}$. Then U is invariant under $\pi(G)$, $\pi(U(\mathfrak{g}))\psi_0 = \bigoplus_{\gamma\in\widehat{K}} U(\gamma)$, and $\dim_{\mathbb{C}} U(\gamma) < \infty$.*

Proof. We know from Lemma 2.19 that U is $\pi(G)$ invariant. Put $U_0 = \pi(U(\mathfrak{g}))\psi_0$. Then $U_0 \subseteq V_K$ and therefore $U_0 = \bigoplus_{\gamma\in\widehat{K}} U_0 \cap V(\gamma)$. Let $\mathfrak{Y}$ be the set of all elements $x \in U(\mathfrak{k})$ such that $\pi(x)\psi_0 = 0$. Then $\mathfrak{Y}$ is a left ideal in $U(\mathfrak{k})$ and it satisfies the conditions prior to

Lemma 2.18. Set $X = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{Y}$. Put $(\tilde{\pi}, X)$ the associated representation of $U(\mathfrak{g})$ on X. Define the map

$$\alpha : X \to U_0 \qquad\qquad \alpha([x]) = \pi(x)\psi_0$$

where x is a representative of $[x]$. This is well defined as for any two representative $x, x' \in [x]$, we have that $x = x' + y$ where $y \in \mathfrak{Y}$. Now $\alpha(x) = \alpha(x') + \alpha(y) = \alpha(x')$ by the definition of $\mathfrak{Y}$. Now, for every $a \in U(\mathfrak{g})$, we have that $\alpha(\tilde{\pi}(a)[b]) = \pi(a)\alpha([b])$. Further, α is injective and thus by identifying X with its image in U_0, can consider $X \subseteq U_0$ and thus we have a direct sum decomposition

$$X = \bigoplus_{\gamma \in \widehat{K}} X(\gamma)$$

and $\alpha(X(\gamma)) = U_0 \cap V(\gamma)$. Applying Lemma 2.18, we get that each $X(\gamma)$ is a finite module over $Z(\mathfrak{g})$. Pick $[b_1], [b_2], ..., [b_k]$ elements in $X(\gamma)$ such that $X(\gamma) = \bigoplus_i \tilde{\pi}(Z(\mathfrak{g}))[b_i]$. Hence, $U_0 \cap V(\gamma) = \bigoplus_i \pi(Z(\mathfrak{g}))\alpha([b_i])$. Now for each element of $z \in Z(\mathfrak{g})$,

$$\pi(z) = \chi(z)\pi(1)$$

and thus the $\alpha([b_i])$ span $U_0 \cap V(\gamma)$ and thus $\dim U_0 \cap V(\gamma) < \infty$. Since $U_0 = \bigoplus_{\gamma \in \widehat{K}} U_0 \cap U(\gamma)$ is dense in U, by Lemma 2.20 we conclude that $U(\gamma) = U_0 \cap U(\gamma)$ and each of these is finite dimensional. This concludes the proof. □

Proof of Theorem 2.17. It is known that every irreducible unitary representation has an infinitesimal character. As the K-finite vectors are analytic and dense in V. Pick $\psi_0 \in V_K$ non-zero. Then by irreducibility, $V = \overline{\pi(U(\mathfrak{g}))\psi_0}$ and applying the above proposition, we conclude that $\dim V_K(\gamma) < \infty$. Hence, every irreducible unitary representation is admissible. □

The remaining part of this section will encompass the proof that irreducibility of admissible representations can be checked on either the group or $(\mathfrak{g}, K)$ level. We will prove this by way of considering the K-finite matrix coefficients for the given representations. In particular, the main tool is actually a fact from the theory of partial differential equations which we quote below:

Theorem 2.23 (Regularity Theorem)**.** *Let D be an elliptic differential operator on $C^\infty(G)$. Then if the coefficients of D are real analytic and u is a solution to $Du = 0$, then u is real analytic.*

Proof. [Gru09, Theorem 6.29]. □

What we shall show is that every K-finite matrix coefficient is annihilated by an elliptic differential operator and thus by the previous theorem, every K-finite matrix coefficient is real analytic.

Proposition 2.24. *Let G be a real reductive group and (π, V) an admissible representation of G. Then every matrix coefficient of the form $g \mapsto (\pi(g)u, v)$ for $u \in V_K$ is real analytic.*

Proof. By unraveling the definitions, we see that for any $D \in U(\mathfrak{g})$, we have that

$$D(\pi(g)u, v) = (\pi(g)\pi(D)u, v)$$

We may assume without loss of generality that u is contained in some $V(\gamma)$. As π is admissible this is finite dimensional and there exists $c_1, ..., c_n \in \mathbb{C}$ such that

$$\prod_i \pi(\Omega) - c_i = 0$$

where Ω is the Casimir element in $Z(\mathfrak{g})$. If we denote by Ω_K the Casimir element of $Z(\mathfrak{k}_{\mathbb{C}})$ then $\pi(\Omega_K) = c_\gamma$ on $V(\gamma)$ by Schur's Lemma. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan Decomposition of the Lie algebra and pick bases for $\mathfrak{k}_0$ and $\mathfrak{p}_0$ which are orthogonal with respect to the inner product $B_\theta(X, Y) = -B(X, \theta Y)$ where θ is the Cartan involution. Then we have the following equalities

$$\begin{aligned}\Omega &= -\sum X_i^2 + \sum Y_i^2 \\ \Omega_K &= -\sum X_i^2 \\ \Omega - 2\Omega_K &= \sum X_i^2 + \sum Y_i^2\end{aligned}$$

This is seen to be an elliptic differential operator by investigating its principal symbol in a neighbourhood of the identity using coordinates defined by the exponential function. From this, we see that the differential operator

$$D = \prod_i \Omega - 2\Omega_K + 2c_\gamma - c_j$$

is also elliptic with real analytic coefficients (as K is an analytic manifold). Now, from our equation above,

$$\begin{aligned}D(\pi(g)u, v) &= \left(\pi(g) \prod_i \left[\pi(\Omega) - \pi(2\Omega_K) + 2c_\gamma - c_j\right] u, v\right) \\ &= \left(\pi(g) \prod_i \left[\pi(\Omega) - c_j\right] u, v\right) \\ &= 0\end{aligned}$$

Hence, by the Regularity Theorem $(\pi(g)u, v)$ is real analytic. □

Definition 2.25. Let (π, V) be a representation of G on a HIlbert (Banach,Fréchet, etc.) space and V_K the associated $(\mathfrak{g}, K)$-module. Then the **contragredient representation** or **dual representation** of V_K is denoted as $V_K^\star$ and is defined to be

$$V_K^\star = (V_K^*)_K$$

There is a natural transpose action on $V_K^\star$ and from this we see that all linear functionals $(-, v)$ for v K-finite are contained in V_K^*. In fact, this is the entire space!

Corollary 2.26. *Let (π, V) and (π', V') be irreducible admissible representations of G. If π and π' are infinitesimally equivalent (V_K and V_K' are isomorphic $(\mathfrak{g}, K)$-modules) then they have the same set of matrix coefficients.*

Proof. The matrix coefficients on G are characterized as the unique real analytic functions such that their derivative at $g = 1$ is given by

$$D(\pi(g)u, v) = (\pi(D)u, v)$$

From the discussion in the definition of the contragredient representation, we see that this is real analytic and given by $(\pi(D)u, v')$ for $v' \in V_K^\star$ not depending on D. Therefore, the matrix coefficients are given in a way which is infinitesimally independent. This completes the proof. $\square$

Corollary 2.27. *The closed G-invariant subspaces of V are in one-to-one correspondence with the $\mathfrak{g}$-invariant subspaces of V_K with the correspondence given by*

$$U \mapsto U \cap V_K$$

and

$$\overline{W} \leftarrow W$$

Proof. This is a particular case of the previous corollary. $\square$

Corollary 2.28. *Let (π, V) be an admissible representation of G. Then V is irreducible if and only if V_K is an irreducible $(\mathfrak{g}, K)$-module.*

Proof. This follows immediately from the previous corollary. $\square$

Corollary 2.29. *Let (π, V) and (π', V') be irreducible admissible representations of G. If V and V' share a single matrix coefficient in common, then $V_K \cong_{\mathfrak{g},K} V'_K$.*

Proof. Our assumption is that

$$(\pi(g)u, v) = (\pi'(g)u', v')$$

for all $g \in G$ and some non-zero $u, v \in V_K$ and $u', v' \in V'_K$. Let $V_0 = U(\mathfrak{g})(\pi(g)u, v)$ be a subspace of $C^\infty(G)$. By the previous theorem, we have that $\pi(U(\mathfrak{g}))u = V_K$. and thus

$$V_0 = (\pi(g)\pi(U(\mathfrak{g}))u, v) = (\pi(g)V_K, v)$$

Define $\varphi : V_K \to V_0$ by $v \mapsto (\pi(-)\pi(U(\mathfrak{g}))u, v)$. This map is onto by construction. Further, it is $U(\mathfrak{g})$-equivariant since

$$\begin{aligned}\varphi(\pi(D)(\pi(D')u)) = \varphi(\pi(DD')u) &= (\pi(-)\pi(DD')u, v) \\ &= D(\pi(-)\pi(D')u, v) \\ &= D\varphi(\pi(D')u)\end{aligned}$$

Since π is irreducible, $\ker \varphi = 0$ and thus it is a $U(\mathfrak{g})$-module isomorphism.

Starting with π' instead, we get a $U(\mathfrak{g})$-module isomorphism $\psi : V'_K \to V_0$ and thus by taking $\psi^{-1}\varphi$ be have the desired $(\mathfrak{g}, K)$-module isomorphism. $\square$

Corollary 2.30. *Consider the induced representations $V = L^2(G, V_\sigma, \sigma), W = C^\infty(G, V_\sigma, \sigma), U = C(G, V_\sigma, \sigma)$ where the latter two are defined naturally. Then $V_K \cong W_K \cong U_K$.*

Proof. Modulo the result that these representations are admissible, we see that for any smooth, L^2 function $G \to V$ we have that the matrix coefficient corresponding to this element will be equal in all of the above representations. Now by the previous corollary, we conclude the result. $\square$

2.2 Infinitesimal Characters

We first recall the notation from above: $\mathfrak{n}, \mathfrak{n}^-$ are the positive and negative root spaces of $\mathfrak{g}$, $Z(\mathfrak{g})$ is the center of the universal enveloping algebra. We can decompose $\mathfrak{g}$ as $\mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}^-$.

Lemma 2.31.

(a) $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n} + \mathfrak{n}^- U(\mathfrak{g}))$

(b) *If* $z \in Z(\mathfrak{g})$ *then* $\pi_2(z) \in U(\mathfrak{g})\mathfrak{n}$ *only.*

Proof. The proof of both parts is a corollary of the Poincaré -Birkhoff-Witt Theorem.

(a) (a) A basis for $U(\mathfrak{g})$ is given by the monomials

$$E_{-\beta_1}^{i_1} ... E_{-\beta_k}^{i_k} H_1^{j_1} ... H_N^{j_n} E_{\beta_1}^{l_1} ... E_{\beta_k}^{l_k}$$

Therefore, identifying $U(\mathfrak{h})$ with the monomials for which $i_p, l_p = 0$ for all p, and the other spaces accordingly, (a) follows.

(b) (b) Let us expand z in terms of the basis given above. Every monomial is an eigenvector for ad H with eigenvalue

$$\sum l_p \beta_p - \sum i_p \beta_p$$

As $z \in Z(\mathfrak{g})$, $(\text{ad}\, H)z = 0$. Therefore if an $E_{-\beta}$ is present, so must be E_β. Therefore $\pi_2(z) \in U(\mathfrak{g})\mathfrak{n}$.

$\square$

Now define $\pi_1 = \gamma' : U(\mathfrak{g}) \to U(\mathfrak{h})$. By Lemma 2.31, we have that for all $z \in Z(\mathfrak{g})$,

$$z - \gamma'(z) \in U(\mathfrak{g})\mathfrak{n} \tag{2}$$

Now let $\lambda \in \mathfrak{h}^*$ and let V_λ be an irreducible representation of highest weight λ. As λ is a linear functional, by the universal property of the symmetric algebra, we can extend λ to a linear function

$$\lambda : S(\mathfrak{h}) \to \mathbb{C}$$

As $\mathfrak{h}$ is a Cartan subalgebra, it is then abelian. Therefore we have an algebra isomorphism and this extends λ as

$$U(\mathfrak{h}) \to S(\mathfrak{h}) \to \mathbb{C}$$

For all $z \in Z(\mathfrak{g})$, we have that

$$zv = \lambda(\gamma'(z))v \qquad \forall v \in V_\lambda$$

Define a map $\sigma : \mathfrak{h} \to U(\mathfrak{h})$ by

$$\sigma(H) = H - \delta(H)1$$

We can again extend this map and it becomes an automorphism of $U(\mathfrak{h})$.

Definition 2.32. The **Harish-Chandra map** is defined as the composition of γ' and σ. That is

$$\gamma : U(\mathfrak{g}) \to U(\mathfrak{h})$$

Remark 2.33. We can relate γ and γ' in the following sense

$$(\lambda + \delta)(\gamma(z)) = \lambda(\gamma'(z)$$

So on V_λ, we have that

$$zv = (\lambda + \delta)(\gamma(z))v$$

Theorem 2.34 (Harish-Chandra)**.** *The Harish-Chandra map is an algebra isomorphism*

$$\gamma : Z(\mathfrak{g}) \to U(\mathfrak{h})^W$$

where $U(\mathfrak{h})^W$ is the subalgebra of elements fixed by the Weyl group action.

Remark 2.35. Every $w \in W$ has a representative in G normalizing $\mathfrak{h}$ under the adjoint map. We extend this action to $U(\mathfrak{h})$. As $\{H_i\}$ is a basis for $\mathfrak{h}$, we have that $U(\mathfrak{h})$ consists of all polynomials in these basis elements. W acts on this space by permuting the indicies.

Before we give a proof of the isomorphism. we need a few lemmas.

Lemma 2.36. *Let $V(\lambda)$ be the verma module for the highest weight λ.*

(a) $V(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda-\delta}$

(b) If $z \in Z(\mathfrak{g})$, then z acts on $V(\lambda)$ as

$$z(u \otimes 1) = \lambda(\gamma(z))(u \otimes 1) \tag{3}$$

Proof. The proof is trivial and will be omitted here. □

Lemma 2.37. *Let $\lambda \in \mathfrak{h}^*, \alpha \in \Pi, m = 2\frac{\langle \lambda, \alpha \rangle}{|\alpha|^2} \geq 0 \in \mathbb{Z}$. Then the $U(\mathfrak{g})$ submodule $M = U(\mathfrak{g})(E_{-\alpha})^m(1 \otimes 1)$ of $V(\lambda)$ is isomorphic to $V(s_\alpha \lambda)$.*

Proof. We shall break this proof into four steps.

Step 1: Put $v = (E_{-\alpha})^m(1 \otimes 1)$. For $H \in \mathfrak{h}$ we have that

$$Hv = (-m\alpha(H) + (\lambda - \delta)(H))v = (s_\alpha \lambda - \delta)(H)v$$

Step 2: We need to show that $E_\beta v = 0$ for all $\beta \in \Delta^+$. It suffices to check for $\beta \in \Pi$ as every positive root can be written as iterated brackets of simple roots. Notice then that $[E_\beta, E_{-\alpha}] = 0$ as $\beta - \alpha$ is not a root for any simple root β. Therefore $E_\beta v = 0$ by definition of highest weight module.

Step 3: We need to show that $E_\alpha v = 0$. We know that

$$E_\alpha v = E_\alpha (E_{-\alpha})^m (1 \otimes 1) = [E_\alpha, E_{-\alpha}^m](1 \otimes 1)$$

Further $[E_\alpha, E_{-\alpha}] = 2|\alpha|^{-2} H_\alpha$. Therefore we have a lie algebra isomorphism

$$\mathrm{Span}_{\mathbb{C}}\{E_\alpha, E_{-\alpha}, H_\alpha\} \cong \mathfrak{sl}_2(\mathbb{C})$$

It can be shown that in $\mathfrak{sl}_2(\mathbb{C})$ we have that

$$[e, f^m] = mf^{m-1}(h - (m-1)1)$$

Using this and normalizing, we see that

$$\begin{aligned}[E'_\alpha, (E'_{-\alpha})^m](1\otimes 1) &= m(E'_{-\alpha})^{m-1}(2|\alpha|^{-2}H_\alpha - (m-1)(1\otimes 1)\\ &= m\left(\frac{2\langle \lambda-\delta,\alpha\rangle}{|\alpha|^2} - (m-1)\right)(E'_{-\alpha})^{m-1}(1\otimes 1)\\ &= 0\end{aligned}$$

The last line follows from the fact that $2\langle \delta, \alpha\rangle / |\alpha|^2 = 1$. This gives us a $U(\mathfrak{b})$ map of $\mathbb{C}_{s_\alpha\lambda-\delta} \hookrightarrow V(\lambda)$.

Step 4: Frobenius Reciprocity gives an isomorphism

$$\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_{s_\alpha\lambda-\delta}, M) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_{s_\alpha\lambda-\delta}), M)$$

This makes the map $1\otimes 1 \mapsto v$ surjective as

$$V(s_\alpha\lambda) = U(\mathfrak{g})(1\otimes 1) \mapsto U(\mathfrak{g})v = M$$

The map is clearly injective and therefore an isomorphism. This completes the proof. □

We now can proceed with the proof of Theorem 2.34. We will stick to the case of $G = U(n), \mathfrak{g}_0 = \mathfrak{u}(n), \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}(n)_{\mathbb{C}}$. The general case follows the same general principles.

Proof of Theorem 2.34 for U(n).

Proof that $\mathrm{Im}(\gamma) \subseteq U(\mathfrak{h})^W$: It suffices to show that

$$s_\alpha\lambda(\gamma(z)) = \lambda(\gamma(z))$$

for each s_α as they generate W. We know by construction that z acts on $v \in V(\lambda)$ by $(\lambda-\delta)(\gamma(z))v$. If we apply s_α we have by the previous lemma that z acts on $V(s_\alpha\lambda)$ by $(s_\alpha\lambda-\delta)(\gamma(z))$. However via the embedding of $V(s_\alpha\lambda) \hookrightarrow V(\lambda)$ given by the lemma, the conclusion follows.

Proof that γ is multiplicative: Recall that $z - \gamma'(z) \in U(\mathfrak{g})\mathfrak{n}$. As σ is an automorphism it suffices to check that γ' is multiplicative. Consider two elements $z_1, z_2 \in Z(\mathfrak{g})$. Then

$$z_1z_2 - \gamma'(z_1)\gamma'(z_2) = z_1(z_2-\gamma'(z_2)) + (z_1 - \gamma'(z_1))\gamma'(z_2)$$

The right hand side is in $U(\mathfrak{g})\mathfrak{n}$ by lemma 2.1. Apply γ' again to the whole expression and we achieve our desired result.

Proof that γ is injective: If $\gamma(z) = 0$ then $\gamma'(z) = 0$ and $\mu(\gamma(z)) = 0$ for all highest wieghts $\mu \in \mathfrak{h}^*$. For any irreducible representation V we know that z acts by the scalar $\lambda(\gamma(z))$ where λ is the highest weight of the representation. As $U(n)$ is reductive, we know that every finite dimensional representation decomposes as a direct sum of irreducible ones. Therefore for any such representation $\Pi : U(n) \to GL(V)$, the associated representation on lie algebras extends to the universal enveloping algebra and therefore $\pi : U(\mathfrak{g}) \to \mathrm{End}(V)$ must have $\pi(z) = 0$.

Now consider the matrix coefficient $(\Pi(g)u, v)$ of Π. It can be shown that for every $D \in U(\mathfrak{g})$,

$$D(\Pi(g)u, v) = (\Pi(g)\pi(D)u, v)$$

Putting $D = z$ we have that z as a left-invariant differential operator, annihilates every matrix coefficient of every finite-dimensional representation of $\mathfrak{g}$. This holds in particular for the tensor powers of the standard representation and therefore z annihilates all monomials and thus $z = 0$ as a differential operator. Hence $z = 0$ in $U(\mathfrak{g})$ and γ is injective.

Proof that γ is onto: Let $\{H_i\}$ be a basis for $\mathfrak{h}$. It is easy to show that $U(\mathfrak{h})^W$ is precisely given by

$$U(\mathfrak{h})^W = \mathbb{C}[s_1, ..., s_n]$$

where s_i is the i^{th} symmetric polynomial in the H_i. Let X_i run through the basis vectors of $\mathfrak{g}$ and put $\tilde{X}_i$ to be the orthogonal basis to X_i with respect to the Killing Form B. Then we can build elements Z_k defined similarly to Casimir elements, which are central and such that

$$\gamma'(Z_k) = \sum_{i_1,...,i_k} Tr(H_{i_1}...H_{i_k})\tilde{H_{i_1}}...\tilde{H_{i_k}} \mod U_{k-1}(\mathfrak{g})$$

. It follows then that $H_i = \tilde{H_i}$ for all i. Therefore

$$\gamma(Z_k) = \sum (H_i)^k \mod U_{k-1}(\mathfrak{g}).$$

Taking the difference between $\gamma(Z_k)$ and $\sum_i H_i^k$ we see that this is a symmetric polynomial. Further, it is polynomial in $p_1, ..., p_{k-1}$. We need to show that p_k is also needed. Let $\zeta_1, ..., \zeta_k$ be the k^{th} roots of unity and consider the homomorphism

$$H_j \mapsto \begin{cases} \zeta_j & 1 \le j \le k \\ 0 & k+1 \le j \le n \end{cases}$$

Under this homomorphism, we have that $p_j(\zeta_1, ..., \zeta_k, 0, ..., 0) = (-1)^{k+1}$ if $j = k$ and 0 otherwise. Further it sends $\sum (H_i)^k$ to n and this implies that $\sum H_i^k$ cannot be expressed in terms of the $p_1, ..., p_{k-1}$ alone. Hence, γ is surjective and this completes the proof of the theorem. □

Proposition 2.38. *Let $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ be an algebra homomorphism sending 1 to 1. Then $\chi = \chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$. Further $\chi_\lambda = \chi_{\lambda'}$ if and only if $\lambda = \lambda' w$ for some $w \in W$.*

In particular, this proposition combined with Dixmier's lemma implies that any irreducible admissible $(\mathfrak{g}, K)$-module V has an infinitesimal character given by χ_λ. Whence, we see that V is annihilated by the ideal $J_\lambda = \langle \ker \chi_\lambda \rangle$ in $U(\mathfrak{g})$.

3 Equivariant Sheaves and the Sheafification of Representations

We now start the transformation of representations into sheaves.

By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g})$ isa filtered algebra and the associated graded ring $\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g})$ is a natural way. In particular, and module over $U(\mathfrak{g})$ inherits a grading and thus we obtain a functor

$$\operatorname{gr} : U(\mathfrak{g})\textbf{-Mod} \to S(\mathfrak{g})\textbf{-Mod}$$

The advantage of this approach is that $S(\mathfrak{g})$ is commutative. In particular, we have a natural identification for all vector spaces that $S(W) = P(W^*)$ polynomial functions on the dual space. Hence, we have the following fact:

Proposition 3.1. $\mathrm{mSpec}(S(\mathfrak{g})) = \mathfrak{g}^*$.

As $S(\mathfrak{g})$ is noetherian, we can invoke Serre's theorem to obtain the following:

Theorem 3.2. *There exists an equivalence of categories between $S(\mathfrak{g})$-$\boldsymbol{Mod}_{fg} \simeq \mathrm{Coh}(\mathfrak{g}^*)$.*

Combining this with the above observation we get a functor (by abusing notation)

$$\mathrm{gr} : \mathcal{M}_{fg}(\mathfrak{g}, K) \to \mathrm{Coh}(\mathfrak{g}^*)$$

This functor lands in a particularly interesting subcategory: the category of *K-equivariant* coherent sheaves.

Definition 3.3. Let G be an S-group scheme, and X an S-scheme together with an action morphism $a : G \times_S X \to X$. A sheaf $\mathscr{F}$ on X is G**-equivariant** if there exists an isomorphism:

$$\theta : a^*\mathscr{F} \to \pi_2^*\mathscr{F}$$

and θ satisfies the following cocycle condition:

$$(\pi_{23}^*\theta) \circ [(1 \times a)^*\theta] = (m \times 1)^*\theta$$

which can be viewed as the following diagram

$$\begin{tikzcd}
 & (m \times 1)^* a^* \mathscr{F} \arrow[r, "(m\times 1)^*\theta"] \arrow[dl, "\sim"'] & (m \times 1)^* \pi_2^* \mathscr{F} \arrow[dr, "\sim"] & \\
(1 \times a)^* a^* \mathscr{F} \arrow[dr, "(1\times a)^*\theta"'] & & & \pi_{23}^* \pi_2^* \mathscr{F} \\
 & (1 \times a)^* \pi_2^* \mathscr{F} \arrow[r, "\sim"'] & \pi_{23}^* a^* \mathscr{F} \arrow[ur, "\pi_{23}^*\theta"'] &
\end{tikzcd}$$

which is to say that θ behaves well with respect to associativity.

When we grade a $(\mathfrak{g}, K)$-module and sheafify it, the K-action on $\mathfrak{g}^*$ induces the structure of a K-equivariant sheaf on $\mathfrak{g}^*$. Therefore, we get a version of Serre's theorem for equivariant sheaves

Theorem 3.4 (K-equivariant Serre)**.** *There is an equivalence of categories*

$$(S(\mathfrak{g}), K)\text{-}\boldsymbol{Mod}_{fg} \simeq Coh^K(\mathfrak{g}^*)$$

The first question we can ask here is what is the support of the sheaf associated to a given finitely generated $(\mathfrak{g}, K)$-module. To answer this, we need to discuss the *nilpotent cone* in $\mathfrak{g}^*$.

3.1 The Nilpotent cone and Nilpotent Orbits

Recall that an element of a Lie algebra is considered nilpotent if $X \in [\mathfrak{g}, \mathfrak{g}]$ and $\mathrm{ad}\, X$ is a nilpotent operator. As we are assuming G to be reductive, there exists a non-degenerate bilinear form ψ on $\mathfrak{g}$ which gives a canonical isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. Thus, we can define nilpotence for functionals.

Definition 3.5 (First approach)**.** An element $\lambda \in \mathfrak{g}^*$ is **nilpotent** if the element $X_\lambda \in \mathfrak{g}$ corresponding to λ under ψ is nilpotent.

This definition is a bit cumbersome as it relies on the definition of ψ. For this reason we give another definition of nilpotence that at first glance seems unrelated.

Definition 3.6 (Second Approach)**.** Let $\lambda \in \mathfrak{g}^*$ and put

$$\mathfrak{g}^*(\lambda) = \{Y \in \mathfrak{g} : \mathrm{ad}^*_Y(\lambda) = 0\}$$

Then λ is **nilpotent** if $\lambda|_{\mathfrak{g}^*(\lambda)} \equiv 0$.

Lemma 3.7. *The two definitions of nilpotent elements are equivalent for a reductive lie algebra.*

Proof. The first definition implies the second one as ψ is non-degenerate for $\mathfrak{g}$ reductive and in particular is a scalar multiple of the Killing form on any simple summand of $[\mathfrak{g}, \mathfrak{g}]$. The reverse direction is a bit more difficult and can be found in [CM93]. □

Definition 3.8. Let G be a real reductive Lie group and $\mathfrak{g}$ its complexified Lie algebra. A **coadjoint orbit** $\mathcal{O}_\lambda$ of G is an orbit of a functional $\lambda \in \mathfrak{g}^*$ under the action given by Ad^*.

Theorem 3.9 (Kostant)**.** *Let $\mathcal{O}_\lambda$ be a coadjoint orbit of G. Then*

$$\mathcal{O}_\lambda \cong G/G(\lambda)$$

where $G(\lambda)$ is the stabilizer of λ in G. Further, there exists a canonical symplectic manifold structure on $\mathcal{O}_\lambda$ induced by the Lie bracket.

Sketch of Proof. The first assertion is immediate. For the second assertion: at $\lambda \in \mathcal{O}_\lambda$ define $\omega_\lambda(X, Y) = \lambda([X, Y])$. For $f = \mathrm{Ad}^*(g)\lambda \in \mathcal{O}_\lambda$ and $Ad(g)X = X_f, \mathrm{Ad}(g)Y = Y_f \in T_f\mathcal{O}_\lambda \cong \mathfrak{g}/\mathfrak{g}^*(\lambda)$ put

$$\omega_f(X_f, Y_f) = f([X_f, Y_f]) = \mathrm{Ad}^*(g)\lambda([\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y]) = \lambda([X, Y])$$

Thus, ω is a G-invariant non-degenerate bilinear form. The fact that it is closed follows from the Jacobi identity. Hence, ω is a symplectic form on $\mathcal{O}_\lambda$. □

Theorem 3.10 (Kostant)**.** *There exist only finitely many coadjoint orbits consisting of nilpotent elements. The **nilpotent cone** $\mathcal{N}^*$ of $\mathfrak{g}^*$ is given by $V((S(\mathfrak{g})^G)^+) = V(\mathrm{gr}\, Z(\mathfrak{g})^+)$.*

We will not prove this theorem, but simply give a corollary:

Corollary 3.11. *The associated graded module of an irreducible $(\mathfrak{g}, K)$-module is supported on the nilpotent cone.*

Proof. As $\mathrm{gr}\, J_\lambda \cap \mathrm{gr}\, Z(g) = \mathrm{gr}\, J_\lambda \cap S(\mathfrak{g})^G$ and $J_\lambda \cap S(\mathfrak{g})^G$ has codimension 1, it follows from the above theorem [CM93] that $V(\mathrm{gr}\, J_\lambda) \subseteq \mathcal{N}^*$. (In particular, J_λ cannot contain elements with constant terms as these will not be G invariant). □

Definition 3.12. An ideal $I \subseteq U(\mathfrak{g})$ is called **primitive** if $I = \mathrm{Ann}(V)$ for an irreducible $U(\mathfrak{g})$-module V. The **associated variety** of I is $\mathcal{V}(I) := V(\mathrm{gr}\, I) \subseteq \mathfrak{g}^*$.

Corollary 3.13. *The associated variety of an irreducible $(\mathfrak{g}, K)$-module is a finite union of nilpotent orbits of G.*

Proof. This follows from Kostant's theorem and the previous corollary. $\square$

A much harder theorem of Borho, Brylinski, and Joseph is as follows:

Theorem 3.14 (Borho, Brylinski, Joseph)**.** *Let I be a primitive ideal. Then $\mathcal{V}(I) = \overline{\mathcal{O}}$ for a single nilpotent orbit.*

Corollary 3.15. *Let M be an irreducible $(\mathfrak{g}, K)$-module. Then $\mathrm{AV}(M) := \mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(M)) = \overline{\mathcal{O}}$ for a single nilpotent orbit. In particular, if $\mathscr{M}$ is the K-equivariant sheaf associated to M, then* $\mathrm{Supp}(\mathscr{M}) = \overline{\mathcal{O}}$

4 An Open Problem

Up until now, the goal has been to use the G action on $\mathfrak{g}^*$ to determine structure of the support of a given K-equivariant coherent sheaf. It was Vogan's insight (see [Vog91]) that we should look at K orbits as well as G orbits. in particular, he showed that:

Theorem 4.1 (Vogan)**.** *Let G be reductive and K a maximal compact subgroup. Then the K-orbits of maximal dimension in a given G orbit $\mathcal{O}$ are Lagrangian submanifolds of $\mathcal{O}$. In particular, there are only finitely many such orbits.*

These K-orbits are related to the minimal K-types of a representation whose associated variety is $\overline{\mathcal{O}}$. In this same vein, we have our idea for a geometric classification of $\widehat{K}$ in terms of representations of G.

Question 4.2. Is there a bijection between $\widehat{K}$ and the set of G-orbits in $\mathcal{N}^*$?

The answer is most certainly no. The right hand side is a finite collection whereas the left side is an infinite collection. However, we have not invoked any of the structure of orbits in this question. In particular, $\mathcal{O}_\lambda = G/G(\lambda)$. Therefore, for any representation (σ, V) of $G(\lambda)$ we can build the bundle $G \times_{G(\lambda)} V \to \mathcal{O}_\lambda$. The set of global continuous or smooth sections of this bundle forms a representation of G under left translation. From this, we can then pick a minimal K-type and maybe this will give us a bijection. In particular, Vogan proved that the set of pairs $\{(\mathcal{O}, \sigma)\}$ consisting of nilpotent G orbits and representations of the stabilizer is a basis for the Grothendieck group $K_0(\mathrm{Coh}_K(\mathfrak{g}^*))$. This led him to give the following conjecture:

Conjecture 4.3 (Vogan)**.** *Let G be a reductive algebraic group defined over $\mathbb{R}$ and K the fixed points of a Cartan involution. There is a bijection*

$$\{(\mathcal{O}, \sigma)\} \to \widehat{K}$$

given by sending the pair to the lowest K-type of the induced representation.

The current progress on this conjecture is as follows:

(a) $GL(n, \mathbb{C}))$ was proven by Achar in his thesis.

(b) G a complex semisimple Lie group was proven by Bezrukavnikov.

All other cases are open.

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