Derived Representation Schemes

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Abstract

We investigate at the derived structure of the derived representation scheme $\mathrm{DRep}_G(\mathbb{G}(\mathbb{RP}^2))$, where $\mathbb{G}$ is the Kan loop group functor. This object parametrizes the representations $\mathbb{G}(\mathbb{RP}^2)$ in G, with G an affine algebraic group over the algebraically closed field k. We prove that representation homology, $HR_*(\mathbb{RP}^2, G)$, vanishes above the dimension of the classical representation variety as well as give a formula for the homology groups in general. For higher genus non-orientable surfaces, we prove some results on vanishing but do not deal with the most general case.

1 Introduction

Let G be an algebraic group over the field k which we assume to be algebraically closed. For any group H we can build the affine scheme $\mathrm{Rep}_G(H)$ which is the moduli space of homomorphisms $\mathrm{Hom}(H, G)$ (the name Rep comes from the fact that these homomorphisms are also called *representations* of the group H in G). We can extend the functor Rep to the category of pointed topological spaces by declaring $\mathrm{Rep}_G(X) := \mathrm{Rep}_G(\pi_1(X, x_0))$. The classical schematic structure of this is well known for certain classes of toplogical spaces. In this paper, we will consider the *derived structure* of this scheme. That is, we can associate to this scheme an object,DRep, in the homotopy category of $\mathbf{sCommAlg}_k$. DRep is an example of a derived scheme and comes equipped with homotopy groups which we will denote by $HR_*(X, G)$.

To understand what these objects are we will first review some material from category theory, homotopical algebra, and algebraic geometry. We will not prove most of the background material but will instead refer the reader to [**?**] for algebraic geometry, and [**?**] and [**?**] for category theory and homotopical algebra respectively.

1.1 Tensor Products and Abelian Categories

Let R be a commutative ring (with 1) and A, B, C three $R-$modules.

Definition-Construction 1.1 (Construction)**.** Let $F(A \times B)$ denote the free R-module generated by $A \times B$. Consider the submodule G which is generated by the relations

$$\begin{aligned}(a, b) &\sim (a, b)\\(a+a', b) &\sim (a, b)+(a', b)\\(a, b+b') &\sim (a, b)+(a, b')\\(ar, b) &\sim (a, rb) \quad r \in R\end{aligned}$$

We define $A \otimes_R B = F(A \times B)/G$. As R is commutative, we have that $A \otimes_R B$ is an $R-$module with multiplication defined by the final relation of G. There is a canonical map

$$\otimes : V \times W \to V \otimes W$$

which sends $(v, w) \mapsto v \otimes w$.

Theorem 1.2 (Universal Property)**.** *For every bilinear map $\varphi : V \times W \to U$ there exists a unique linear map $\hat{\varphi} : V \otimes W \to U$ such that $\varphi = \hat{\varphi} \circ \otimes$*

Consider an exact sequence of $R-$modules

$$0 \to L \to M \to N \to 0$$

Using A, we naturally get four functors which come in two pairs

$$\begin{matrix} - \otimes_R A \\ A \otimes_R - \\ \mathrm{Hom}_R(A, -) \\ \mathrm{Hom}_R(-, A) \end{matrix} : \mathbf{Mod}_R \to \mathbf{Mod}_R$$

The first pair are covariant and as R is commutative, $A \otimes_R - \cong - \otimes_R A$ as functors.

Definition 1.3. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is **abelian** if the following are true:

a) $\mathscr{C}$ has a zero object

b) $\mathscr{C}$ is pre-additive (i.e. Hom-sets are abelian groups and composition is bilinear)

c) $\mathscr{C}$ has all finite products and coproducts

d) Every morphism has a kernel and cokernel

e) To any morphism $f : A \to B$ the morphism $p : \mathrm{coim}\, f \to \mathrm{Im}\, f$ is an isomorphism.

Abelian categories are the general setting for exact sequences as we can always form them by property (d).

Definition 1.4. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $F : \mathscr{A} \to \mathscr{B}$ a covariant additive functor. We say that F is **left exact** if for all exact sequences

$$0 \to L \to M \to N \to 0 \implies 0 \to F(L) \to F(M) \to F(N)$$

is exact. We define right exact similarly. For a contravariant functor we have the analogous exact sequences.

It is an easy exercise to show that $- \otimes_R A$ and both versions of Hom are additive functors. It can be proven that $\mathbf{Mod}_R$ is an abelian category. This then gives us the correct context to prove the following theorem.

Theorem 1.5. *$- \otimes_R A$ is right exact while $Hom_R(A, -)$ and $Hom_R(-, A)$ are left exact.*

Theorem 1.6. *In more generality, suppose $(F \dashv G) : \mathscr{A} \leftrightarrows \mathscr{B}$ is an adjunction (i.e. $Hom_{\mathscr{B}}(A, GB) \cong Hom_{\mathscr{A}}(FA, B)$) of additive functors. Then F is right exact and G is left exact.*

Example 1.7. Here is an example to show how $\otimes$ can fail to be left exact. Let $\mathbb{Z}_2$ denote the cyclic group of order 2. Treated as a $\mathbb{Z}$-module, we have $- \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

Applying the functor, we get

$$\mathbb{Z}_2 \to 0 \to \mathbb{Q}/\mathbb{Z} \to 0$$

The reason that $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ follows from the fact that for any element

$$a \otimes b = a \otimes 2b/2 = 2a \otimes b/2 = 0 \otimes b/2 = 0$$

This theorem is the motivation behind most of elementary homological algebra. Notice that neither functor is totally exact but only left or right. We want to find a remedy to this which preserves the original structure.

Definition 1.8. Let M be an element of an abelian category $\mathscr{A}$. A **projective**(resp. injective, free) **resolution** for M is an exact sequence

$$P_\bullet = ... \to P_2 \to P_1 \to P_0 \to M \to 0$$

with each P_i a projective(resp. injective, free) object in $\mathscr{A}$.

Remark 1.9. For injective resolutions, we require that the arrows all be reversed as this reflects the property of the map $M \to I_0$ in the resolution.

Definition 1.10. A **chain complex** in an abelian category $\mathscr{A}$, is an object $(A_\bullet, d_\bullet)$ such that

$$A_\bullet = \;\; ... \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \xrightarrow{d_{i-1}} ...$$

with $d_{i-1} \circ d_i = 0$. We call $d_\bullet$ the **differential of** $A_\bullet$. A morphism or **chain map** of complexes is a map $f : A_\bullet \to B_\bullet$ such that $f_i : A_i \to B_i$ commutes with the differential for each i.

From the differential condition, we know that $\ker d_i \supset \mathrm{Im}\, d_{i+1}$. Therefore, we define the **i-th homology group**

$$H_i(A_\bullet) = \ker d_i / \mathrm{Im}\, d_{i+1}$$

Example 1.11. Consider the bounded chain complex

$$Q = 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$$

where the map is given by multiplication by n. This complex has homology $H_0(Q) = \mathbb{Z}_n$ and $H_1(Q) = 0$.

Remark 1.12. We say that a chain complex is **formal** if there is a chain map $f : A_\bullet \to H_i(A_\bullet)$ which induces an isomorphism on homology. This condition is generalized by **quasi-isomorphism** (i.e. a chain map which induces an isomorphism on homology).

We can measure how $- \otimes_R A$ fails to be left exact in the following way: let M be any $R-$module and $P_\bullet$ a projective resolution of M. Applying $- \otimes_R A$ to $P_\bullet$, we get the sequence

$$P_\bullet \otimes_R A = ... \to P_1 \otimes_R A \to P_0 \otimes_R A \to M \otimes_R A \to 0$$

By Theorem 2.5, we have that $P_0 \otimes_R A \to M \otimes_R A \to 0$ is exact. We see that As $P_\bullet$ is exact, we have that $P_\bullet^+ = ... \to P_0 \to 0$ is quasi-isomorphic to M treated as a chain complex with M in the $0^t h$ degree position. Therefore, truncating at M, we recover the tensor product in homology, but now we get additional higher order homology groups. So, when we take homology of $P_\bullet^+ \otimes_R A$ we define

$$\mathrm{Tor}_i^R(M, A) := H_i(P_\bullet^+ \otimes_R A)$$

Notice that we always have $\mathrm{Tor}_0^R(M, A) = M \otimes_R A$. These are the left derived functors of $- \otimes_R A$. The following proposition

Proposition 1.13. *$Tor_i^R(M, A)$ does not depend on the choice of projective resolution.*

The proof of this proposition makes use of chain homotopies. We will not discuss them here. For the interested reader we refer to [**?**] for a detailed exposition.

The final theorem of this section gives some important properties of Tor.

Theorem 1.14.

a) Let A, B be isomorphic $R-$modules. For any module M and a projective resolution $P_\bullet$ of M. We have that $Tor_i^R(M, A) \cong Tor_i^R(M, B)$ for all i.

b) $Tor_i^R(M, A) = 0$ for all $i \geq 1$ if and only if A is flat.

c) If $(M_\alpha, \varphi^\alpha_\beta)_{\alpha \preceq \beta}$ is a direct system of $R-$modules. Then

$$Tor_i^R \left(\varinjlim M_\alpha, A \right) \cong \varinjlim Tor_i^R(M_\alpha, A)$$

d) If $0 \to L \to M \to N \to 0$ is an exact sequence of $R-$modules, then there is a long exact sequence

$$... \to Tor_i^R(L, A) \to Tor_i^R(M, A) \to Tor_i^R(N, A) \to Tor_{i-1}^R(L, A) \to ...$$

The final property above is precisely the remedy for exactness. Even though the sequence is now a long, as opposed to short, we have preserved exactness.

1.2 Homotopy Categories

In topology, we have the notion of weak homotopy equivalent spaces. For chain complexes there is an analogous notion of quasi-isomorphism. These notions are "the same" in the sense that they represent the same type of map but in different categories. This in general is the idea of **weak equivalence**.

Remark 1.15. We want to have weak equivalences in simplicial sets. To any simplicial set we can associate homotopy groups

$$\pi_* X := \pi_* |X|$$

where $|X|$ is the geometric realization of the simplicial object. Using Dold-Kan, we can extend the notion of homotopy groups to any simplicial category. For example, given a simplical commutative algebra A, we define

$$\pi_*(A) := H_*[N(A)]$$

Maps which induce isomorphisms on homotopy groups are the weak equivalences in **sSet**.

The most important simplicial category for our purposes is **sGrp** the category of simplicial groups. We have the following theorem due to Kan in [**?**].

Theorem 1.16 (Kan)**.** *There is an equivalence of categories given by the adjunction*

$$\mathbb{G} : \boldsymbol{sSet}_0 \rightleftarrows \boldsymbol{sGrp} : \overline{W}$$

*where $\boldsymbol{sSet}_0$ is the category of reduced simplicial sets and $\mathbb{G}$ is the Kan Loop group functor as defined in [**?**].*

Later in the paper we will use $\mathbb{G}$ to define $\mathrm{DRep}_G(\mathbb{G}X)$ for any topological space X. For now, we have a few more ideas that will in turn yield the construction $\mathrm{DRep}_G(-)$.

The categories $\mathbf{s}\mathscr{C}$ are all examples of model categories. These objects are integral to homotopy theory and homotopical algebra. We will give a more informal definition here and refer the reader to [**?**] for more detail.

Definition 1.17. A model category $\mathscr{M}$ is a category equipped with three classes of morphisms: Fibrations, Cofibrations, and Weak Equivalences. These must satisfy a myriad of axioms which specify how these classes of morphisms interact with each other. See [?, Definition 1.0] for the axioms. Just as we have the notion of adjunction between standard categories, between model categories we can have **Quillen adjunctions**. These are adjunctions $(L \dashv R)$ such that L preserves cofibrations and R preserves fibrations.

Example 1.18. We have already encountered an example of a model category. Let $\mathscr{A}$ be an abelian category and $\mathbf{Ch}^+(\mathscr{A})$ be the category of connective(bounded below) chain complexes. The fibrations are epimorphisms, cofibrations are monomorphisms, $i : A \to B$, such that $[\text{coker } i]_n$ is a projective in $\mathscr{A}$. Weak equivalences are quasi-isomorphisms. The Dold-Kan correspondence [?] gives a bidirectional equivalence of categories which is given in one direction by $N : \mathbf{s}\mathscr{A} \to \mathbf{Ch}^+(\mathscr{A})$. Therefore we have that $\mathbf{s}\mathscr{A}$ is also a model category.

Definition 1.19. Let $\mathscr{C}$ be a model category. Let W be the class of weak equivalences. Then we define the **Homotopy Category** to be

$$\mathbf{Ho}(\mathscr{C}) := W^{-1}\mathscr{C}$$

together with a functor $\ell : \mathscr{C} \to \mathbf{Ho}(\mathscr{C})$ satisfying the universal property: given any functor $F : \mathscr{C} \to \mathscr{B}$ between model categories. If $F(w)$ is an isomorphism for all $w \in W$, then there exists a unique functor θ such that the following diagram commutates

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "\ell"] \arrow[d, "F"'] & \mathbf{Ho}\mathscr{C} \arrow[dl, dashed, "\theta"] \\
\mathscr{B} &
\end{tikzcd}$$

$W^{-1}\mathscr{C}$ is the localization of the category $\mathscr{C}$. That is, if $w \in W$ then $\ell(w)$ is an isomorphism in $\mathbf{Ho}\mathscr{C}$.

Remark 1.20. We can restrict $\mathscr{C}$ to the subcategories $\mathscr{C}_c, \mathscr{C}_f$ of cofibrant and fibrant objects respectively. Suppose we have a **homotopical functor** $F : \mathscr{C} \to \mathscr{D}$ (this is a functor between model categories which preserves weak equivalences). Then it is clear that F induces a functor between the homotopy categories. Further it is easy to show that there is an equivalence of categories

$$\mathbf{Ho}(\mathscr{C}) \cong \mathbf{Ho}(\mathscr{C}_c) \cong \mathbf{Ho}(\mathscr{C}_f)$$

Definition 1.21. If $F : \mathscr{C} \to \mathscr{D}$ is a functor such that F_c the restriction of F to $\mathscr{C}_c$ becomes a homotopical functor, then the **total left derived functor** of F is

$$\mathbf{L}F : \mathbf{Ho}(\mathscr{C}_c) \to \mathbf{Ho}(\mathscr{D})$$

We define the total right derived functor similarly but instead with the full subcategory of fibrant objects.

In short, we will use the notion of total left derived functor to define DRep_G as a functor between **sGrp** and $\mathbf{sCommAlg}_R$. Before we can do this however, we need the language of schemes to give a precise definition of the variety $\mathrm{Rep}_G(\Gamma)$.

1.3 Sheaves and Schemes

There are many standard references which give introductions to algebraic geometry. We modeled this exposition after [**?**] and [**?**]. Before we can tackle schemes, we need the idea of a ringed space and therefore sheaves.

Definition 1.22. Let X be a topological space and $\mathscr{C}$ a category. A **presheaf** on X is a functor

$$\mathcal{F} : \text{Open(X)}^{op} \to \mathscr{C}.$$

$\mathrm{Open}(X)$ is the category whose objects are open subsets of X and whose morphisms are one point sets if $V \subseteq U$. Morphisms of presheaves are natural transformations of functors.

Remark 1.23. Notice that for $V \subseteq U$, there is a unique morphism denoted $\mathrm{Res}^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$. We sometimes call $\mathcal{F}(U)$ the set of sections of $\mathcal{F}$ over U, and denote this $\Gamma(U, \mathcal{F})$. Additionally, instead of writing $\mathrm{Res}^U_V(s)$ for the image of s in $\mathcal{F}(V)$, we instead write $s|_V$.

Every presheaf is the same as a contravariant functor. We use the term presheaf when we want to discuss some gluing conditions which we will see later. Some classical examples of presheaves are

$$C^\alpha_M = \{f : M \to \mathbb{R} : f \text{ is } \alpha \text{ times differentiable}\}$$

for a real manifold M and $\alpha \in \mathbb{N} \cup \{\infty\}$.

Definition 1.24. Let X be a topological space and $\mathcal{F}$ a presheaf on X. $\mathcal{F}$ is a **sheaf** if the following condition is satisfied

(Sh) If $U \subseteq X$ is an open set and $\{U_i\}_{i \in I}$ is an open cover of U such that for all i there exists $f_i \in \mathcal{F}(U_i)$ and for all $i \neq j \in I$ $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ then there exists a unique $f \in \mathcal{F}(U)$ such that $f_{U_i} = f_i$.

Remark 1.25. This definition can be generalized to categories. To do this correctly however one needs the language of sites. We will not cover these here but refer the reader to [**?**] for the case of site and stacks in the smooth category and [**?**] for the more categorical treatment.

Example 1.26. We have already seen an example of a sheaf, namely C^α. It is easy to check the gluing condition (Sh). Other common examples are Ω^p_M the set sheaf of differential forms of degree p and L the sheaf of locally constant functions on a space.

Definition 1.27. Let X be a topological space and $U(x) = \{U \in \text{Open}(X) : x \in U\}$. Suppose $\mathcal{F}$ is a presheaf on X. We define the **stalk** of $\mathcal{F}$ to be

$$\mathcal{F}_x = \varinjlim_{U(x)} \mathcal{F}(U)$$

Proposition 1.28. *A morphism of sheaves on a space X, $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if it the induced map on stalks $\varphi : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism.*

The collection of all $\mathscr{C}-$valued sheaves on a topological space for a category denoted $Sh(X, \mathscr{C})$ (Presheaves also form a category). Notice that for a morphism of sheaves with values in an abelian category, the kernel presheaf defines a sheaf but the cokernel presheaf does not. Further, we would like for quotients to exist in this category. To remedy this, we come to the following definition.

Definition-Proposition 1.29. For any presheaf $\mathcal{F}$ there is a sheaf $\mathcal{F}^+$ and a natural morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ with the following universal property: for any sheaf $\mathcal{G}$ and morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there exsits a unique morphism of sheaves $\widehat{\varphi} : \mathcal{F}^+ \to \mathcal{G}$ with $\widehat{\varphi} \circ \theta = \varphi$. That is, the following diagram commutes

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[d, "\varphi"'] & \mathcal{F}^+ \arrow[dl, dashed, "\widehat{\varphi}"] \\ \mathcal{G} & \end{tikzcd}$$

The sheaf $\mathcal{F}^+$ is called the **sheafification** of $\mathcal{F}$. One can prove that sheafification is functorial in presheaves.

Remark 1.30. There is another way to build the sheaf associated to a presheaf. Given a presheaf $\mathcal{F}$ on X, we can construct a sheaf $\textbf{Spé}(\mathcal{F}) = \bigsqcup_{p \in X} \mathcal{F}_p$. This has a natural projection $\pi : \textbf{Spé}(\mathcal{F}) \to X$ which projects each stalk onto the point it is over. We topologize this space by endowing it with the strongest topology such that the sections $s \in \mathcal{F}(U)$ are continuous. Sections of the map π are continuous. It can be shown that these definitions agree.

The sheafification operation allows us to define cokernels, quotients, and constant sheaves. All of this together tells us that if $\mathscr{A}$ is an abelian category, then $\text{Sh}(X, \mathscr{A})$ is also an abelian category. Up until this point, we have considered a fixed space X. If we have a morphism of topological spaces $f : X \to Y$, we want to build a sheaf on Y which comes from f in some way.

Definition 1.31. Let $f : X \to Y$ be a map of topological spaces. Suppose $\mathcal{F}$ is a sheaf on X. The **direct Image** (or **pushforward**) sheaf on Y with respect to f is the sheaf

$$f_* \mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$$

Further, we define the **inverse image** sheaf on X of a sheaf on Y as

$$f^{-1}\mathcal{G}(U) = \varinjlim_{f(U) \subset V} \mathcal{G}(V)$$

Remark 1.32. In the previous definition, one may want to give a naive definition of the inverse image sheaf in the the style of the pushforward, that is $f^{-1}\mathcal{G}(U) = \mathcal{G}(f(U))$. This fails immediately however as we are not guaranteed that $f(U)$ is open.

Sometimes, topological spaces come naturally equipped with sheaves. Examples of this situation are smooth manifolds. to every real topological manifold M, we have C^0_M the sheaf of continuous functions $M :\to \mathbb{R}$. Another example is schemes which we will cover shortly.

Definition 1.33. A **ringed space** is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$ called the structure sheaf of X. A morphism of ringed spaces is a pair $(f, f^\sharp)$ with $f : X \to Y$ a continuous map and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ a map of sheaves. We call $(X, \mathcal{O}_X)$ a locally ringed space if the stalks $\mathcal{O}_{X,p}$ are local rings for all $p \in X$. A morphism of locally ringed spaces is a pair where the map on sheaves is a local homomorphism of local rings (on stalks it sends the maximal ideal at $f(p)$ to the maximal ideal at p surjectively).

For the rest of this section we will concern ourselves with the topological space $\mathbf{Spec}(A) = \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a prime ideal}\}$ associated to the ring A. We will not review the definition of the Zariski topology and refer the reader to [**?**] for details. This space comes naturally equipped with a sheaf of rings $\mathcal{O}_{\mathbf{Spec}(A)}$. Generally speaking $\Gamma(\mathbf{Spec}(A), \mathcal{O}_{\mathbf{Spec}(A)})$ is called the collection of **global sections**.

Definition 1.34. An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $\mathbf{Spec}(A)$ for some ring A. Affine schemes form a category with morphisms being morphism of locally ringed spaces. In general

$$\mathbf{Spec}(-) : \mathbf{CommRing} \to \mathbf{Aff}$$

is a fully faithful functor and therefore defines an equivalence of categories.

Remark 1.35. We will not give the definition of a general scheme but will instead give an analogy with manifolds. Similar to how manifolds are determined by coordinate charts which give local diffeomorphism to subsets of $\mathbb{R}^n$, for schemes, every point in a general scheme is contained in an open set U which is isomorphic to $\mathbf{Spec}(A)$ for some commutative ring A.

Definition 1.36. An **affine algebraic group** (over the field k) is a scheme G together with three morphisms

a) $\mu : G \times_{\mathbf{Spec}\, k} G \to G$

b) $\varepsilon : \mathbf{Spec}\, k \to G$

c) $i : G \to G$

satisfying the unsual group relations. This is also called a group scheme. Note that algebriac groups are group objects in **Aff**.

Example 1.37. The most common example of an algebraic group is $GL_n(k)$. In fact, we can view GL_n as a group scheme and $Gl_n(k)$ as its $k-$points. Other examples are $SL_n, SO(n, \mathbb{C}), SU(n), etc$. Nearly all matrix groups can be realized as schemes. In the literature it is standard to see the groups $GL_1(k) = k^\times$ and $M_1(k) = k$ denoted as $\mathbb{G}_m$ and $\mathbb{G}_a$ respectively.

The final idea we will discuss is sheaves of modules over the structure sheaf of a scheme. These objects will be integral to the discussions in later sections of the paper.

Definition 1.38. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X-$module over X is a sheaf $\mathcal{F}$ such that for each open set $U \subseteq X$, we have that $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module. A morphism of $\mathcal{O}_X-$modules is a morphism of sheaves so that the resulting map on open sets is an $\mathcal{O}_X(U)-$module map.

Now that we have the notion of a scheme, we have enough tools to finally give a description of $\mathrm{DRep}_G(\mathbb{G}(\mathbb{RP}^g))$. We will follow the discussion in [**?**]. Fix an algebraic group G. For any group Γ we can consider the space of all representaitons (i.e. group homomorphisms $\Gamma \to G$) up to isomorphism. This space is denoted $\mathrm{Rep}_G(\Gamma)$. It can be realized as a scheme and therefore as a derived scheme.

To build the corresponding derived scheme however, we need another functor. The group G induces a functor $(-)_G : \mathbf{Grp} \to \mathbf{CommAlg}_k$. If Γ is a group, then

$$\Gamma_G = Sym_k^*(k[\Gamma] \otimes_k \mathcal{O}(G))/\langle\langle R\rangle\rangle$$

where R is defined in [**?**]. This extends to a homotopical functor on the associated simplicial categories ([**?**][Lemma 4.1]. Therefore by definition 2.28, there exists a total left derived functor

$$\mathbf{L}(-)_G : \mathbf{Ho}(\mathbf{sGrp}) \to \mathbf{Ho}(\mathbf{sCommAlg}_R)$$

We define $\mathrm{DRep}_G(-) := \mathbf{L}(-)_G$. This is the derived representation scheme of over the group G. For a group Γ with n generators one relation r, we can realize this as the fibre product

$$\begin{tikzcd} \mathrm{DRep} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G^n \arrow[d] \\ * \arrow[r] & G \end{tikzcd}$$

where the map $G^n \to G$ is given by $(g_1, ..., g_n) \mapsto r(g_1, ..., g_n)$. Further, as shown in [**?**, Proposition 4.2] we have the following isomorphism

$$HR_*(X, G) \cong \pi_*[\mathrm{DRep}_G(\mathbb{G}X)]$$

For the case of $X = \mathcal{N}_g$ the non-orientable surface of genus g, it can be shown that

$$\mathrm{DRep}_G(\mathbb{G}X) \simeq \mathcal{O}(G^g) \otimes^{\mathbf{L}}_{\mathcal{O}(G)} R$$

where $\mathcal{O}(G^g)$ has its module structure given by the map on algebraic groups $G^g \to G$ defined by $(x_1, ..., x_g) \mapsto \prod x_i^2$ and $\otimes^{\mathbf{L}}_{\mathcal{O}(G)}$ is the total left derived functor on the homotopy category of $\mathcal{O}(G)$-modules. From this it is easy to show that

$$HR_*(\mathbb{RP}^2, G) \cong \operatorname{Tor}_i^{\mathcal{O}(G)}(\mathcal{O}(G^g), R)$$

For the remainder of this paper, we will compute bounds for the vanishing of representation homology as well as give explicit descriptions of the groups themselves for certain G.

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