Derived Categories

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Introduction

One of the important theorems of classical commutative algebra is as follows:

Theorem 0.1 (Auslander-Buchsbaum-Serre)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. The following are equivalent:*

(a) *R is regular.*

(b) $\operatorname{projdim}(M) < \infty$ *for all finite R-modules M*

(c) $\operatorname{projdim}(k) < \infty$

This theorem gives a way to check regularity of a ring in terms of its modules. In the more modern language, we can derive a similar result using derived categories and will prove the following theorem:

Theorem 0.2. *Let R be a neotherian ring. Then R is regular if and only if* $\boldsymbol{D}_{sg}(R\text{-}\boldsymbol{Mod}) \simeq 0$.

In particular, this gives a simple answer to the following

Conjecture 0.3. *If R is regular, then any localization of R is regular.*

1 From Abelian categories to Derived Categories

We start by recalling the definition of an Abelian category:

Definition 1.1. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is **abelian** if the following are true:

(a) $\mathscr{C}$ has a zero object

(b) $\mathscr{C}$ is pre-additive (i.e. Hom-sets are abelian groups and composition is bilinear)

(c) $\mathscr{C}$ has all finite products and coproducts

(d) Every morphism has a kernel and cokernel

(e) To any morphism $f : A \to B$ the morphism $p : \operatorname{coim} f \to \operatorname{Im} f$ is an isomorphism.

Abelian categories are the general setting for exact sequences as we can always form them by property (d).

Definition 1.2. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $F : \mathscr{A} \to \mathscr{B}$ a covariant additive functor. We say that F is **left exact** if for all exact sequences

$$0 \to L \to M \to N \to 0 \implies 0 \to F(L) \to F(M) \to F(N)$$

is exact. We define right exact similarly. For a contravariant functor we have the analogous exact sequences.

It is an easy exercise to show that $- \otimes_R A$ and both versions of Hom are additive functors. It can be proven that $\mathbf{Mod}_R$ is an abelian category. This then gives us the correct context to prove the following theorem.

Theorem 1.3. *$- \otimes_R A$ is right exact while $\mathrm{Hom}_R(A, -)$ and $\mathrm{Hom}_R(-, A)$ are left exact.*

More precisely, this following from the following proposition:

Proposition 1.4. *In more generality, suppose $(F \dashv G) : \mathscr{A} \leftrightarrows \mathscr{B}$ is an adjunction (i.e. $\mathrm{Hom}_{\mathscr{B}}(A, GB) \cong \mathrm{Hom}_{\mathscr{A}}(FA, B)$) of additive functors. Then F is right exact and G is left exact.*

Example 1.5. Here is an example to show how $\otimes$ can fail to be left exact. Let $\mathbb{Z}_2$ denote the cyclic group of order 2. Treated as a $\mathbb{Z}$-module, we have $- \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

Applying the functor, we get

$$\mathbb{Z}_2 \to 0 \to \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to 0$$

The reason that $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ follows from the fact that for any element

$$a \otimes b = a \otimes 2b/2 = 2a \otimes b/2 = 0 \otimes b/2 = 0$$

This theorem is the motivation behind most of elementary homological algebra. Notice that neither functor is totally exact but only left or right. We want to find a remedy to this which preserves the original structure. To do this, we pass to a larger category of objects, where we can fully capture this subtlety in a purely algebraic way. For a general abelian category, we have the notion of short exact sequences. In addition to this, we have the notion of *(co)chain complexes*. These will be the central objects we want to consider when answering the questions posed in the previous section.

Definition 1.6. Let $(C_\bullet, d_\bullet)$ be a collection of objects in an abelian category $\mathscr{A}$ together with a morphism $d_n : C_n \to C_{n-1}$. We call $(C_\bullet, d)$ a **chain complex** if $d_{n-1} \circ d_n = 0$. If instead we have an object $(C^\bullet, \partial^\bullet)$ such that $\partial^n : C^n \to C^{n+1}$ such that $\partial^{n+1} \circ \partial^n = 0$ then we say the pair is a **cochain complex**. It is common practice to drop the index on the differential $d_\bullet$ or $\partial^\bullet$ and simply denote them d and ∂. We shall adopt this convention.

A morphism of (co)chain complexes $(C_\bullet, d)$ and $(D_\bullet, d')$ is a **chain map** $f_\bullet$ (resp. $f^\bullet$), that is a collection of maps f_i so that the following diagram commutes for all n,

$$\begin{tikzcd}
C_n \arrow[r, "d"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_n \arrow[r, "d'"'] & D_{n-1}
\end{tikzcd}$$

With this notion of morphism, we can build a new category $(\mathbf{c})\mathbf{Ch}(\mathscr{A})$ of (co)chain complexes. Notice that because of the condition $d^2 = 0$, we have that $\mathrm{Im}\, d_n \subseteq \ker d_{n-1}$.

Definition 1.7. Let $(C_\bullet, d)$ be a chain complex. Define the $n-$**th homology groups** of $C_\bullet$ as

$$H_n(C_\bullet) = \ker d_n / \operatorname{Im} d_{n+1}$$

These are in fact groups.

Two chain complexes are **quasi-isomorphic** if there exists a chain map $f_\bullet : C_\bullet \to D_\bullet$ such that $(f_i)_* : H_i(C_\bullet) \xrightarrow{\sim} H_i(D_\bullet)$ where $(f_i)_*$ is defined as $[\alpha] \mapsto [f_i \circ \alpha]$. This is well-defined by the definition of a chain map. Further $f_i \circ \alpha \in \ker d'_n$. We have completely analogously the definition of **cohomology groups** $H^i(C^\bullet)$. We call a (co)chain complex is *exact* if all of the (co)homology groups are identiically 0.

We now return to the content of the previous section. Let M be an R-module and $P_\bullet$ a projective resolution of M.

It then follows from the discussion above that $P_\bullet \to 0$ (truncating the free resolution of M at P_0) and M are quasi-isomorphic as chain complexes (here M is considered as the trivial chain complex with differential 0 everywhere). We can use this to our advantage. For any R-module A, consider $\operatorname{Hom}(-, A)$. The resulting cochain complex

$$\operatorname{Hom}(P_0, A) \to \operatorname{Hom}(P_1, A) \to \operatorname{Hom}(P_2, A) \to ...$$

is no longer exact.

Definition 1.8. The **n-th cohomology groups** or n-th Ext groups of M and are denoted

$$\operatorname{Ext}^n_R(M, A) := H^i(\operatorname{Hom}(P_n, A))$$

Remark 1.9. It can be shown that these groups do not depend on the resolution taken. In fact, it does not even matter if we resolve A or M. There is a dual construction of $\operatorname{Ext}^n(M, A)$ where instead of a projective resolution of M, we take an injective resolution of A.

For $\otimes$, we have the corresponding construction but now we only use projective resolutions as $\otimes$ is covariant in both arguments.

Definition 1.10. The **n-th homology groups** or n-th Tor groups of M are

$$\operatorname{Tor}^R_n(M, A) := H_i(P_n \otimes A)$$

We now generalize to arbitrary abelian categories.

Definition 1.11. An abelian category is said to **have enough projectives** if every element has a projective resolution (respectively, enough injectives and enough flats)

Let $\mathscr{A}$ be an abelian category with enough projectives and $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor. Then for any projective resolution of an object M, we can repeat the operation above to define the *derived functors* of F. To be more specific, let $P_\bullet$ be a projective resolution of M.

Definition 1.12. The functors

$$L_iF(M) = \ker(FP_n \to FP_{n-1}) / \operatorname{Im}(FP_{n+1} \to FP_n)$$

are called the **left derived functors** of F. Dually if G is left exact and $I^\bullet$ is an injective resolution, we can define R^iG as the **right derived functors** for G.

One may ask why we do not consider the left derived functors for a left exact functor. The answer to this is that these are all zero, or at least un interesting. They tell you nothing about exactness as 0s appear in the sequences.

Proposition 1.13. *If F is exact then R^iF and L_iF are 0 for all $i > 0$.*

Proof. As F is exact, the resulting long sequences are exact. Hence, the quotient groups are 0 and R^iF (resp. L_iF) is 0. □

Remark 1.14. The derived functors measure the extent to which M is not projective, injective, or flat. More generally, they measure how far F is from being exact. If R^iF is non-zero for only very large i, then F is very close to being exact. Whereas if R^2F is non-zero, then F is nowhere close to being exact.

The final theorem we present in this section is the most useful for computing these functors.

Theorem 1.15. *Let $0 \to A \to B \to C \to 0$ be exact in $\mathscr{A}$ and $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor. Then there is a long exact sequence*

$$...L_iF(A) \to L_iF(B) \to L_iF(C) \to L_{i-1}F(A) \to ...$$

in the derived functors. The same is true for left exact functors.

The proof of this is immediate from the Snake Lemma. The reason it is so important is because if we know that either A, B, or C is F-acyclic (that is $L_iF(C) = 0$) then we get isomorphisms of the remaining groups!

1.1 Localization and Chain homotopies

Now that we have the categories $(\mathbf{c})\mathbf{Ch}(\mathscr{A})$ and the notions of quasi-isomorphisms, we can quickly construct the derived category $\mathbf{D}^*(A)$ where $*$ will be a condition on the elements of $\mathbf{D}(A)$. The construction proceeds in two steps involving localization of categories.

Definition 1.16. Let $\mathscr{A}$ and $\mathscr{B}$ be (small) categories[1] such that $\mathscr{B} \subseteq \mathscr{A}$ is a subcategory. Then the **localization** of $\mathscr{A}$ by $\mathscr{B}$ is denoted $\mathscr{B}^{-1}\mathscr{A}$ and is the category who's objects are objects of A but whos morphisms are morphisms in $\mathscr{A}$ together with formal inverses for all morphisms in $\mathscr{B}$. In general, we can localize with respect to *any* collection of morphisms in $\mathscr{A}$.

In particular, this definition is too general to preserve any nice structure of the original category $\mathscr{A}$. Therefore, we have the following related definition for the situations we shall frequently run into:

Definition 1.17. Let S be a class of morphisms in $\mathscr{A}$. We call S a **localizing class** if the following hold:

(a) $1_M \in S$ for all $M \in \mathscr{A}$.

(b) For all $s, t \in S$, $t \circ s \in S$.

[1] Small here is to avoid some set theoretic nonsense, but is not an actual blockade in the definition

(c) For any given morphisms $f \in \operatorname{Mor} \mathscr{A}$ and $s \in S$ there exist $g \in \operatorname{Mor} \mathscr{A}$ and $t \in S$ such that the following diagram commutes

$$\begin{tikzcd} K \arrow[r, dotted, "t"] \arrow[d, dotted, "g"'] & L \arrow[d, "f"] \\ M \arrow[r, "s"'] & N \end{tikzcd}$$

(d) The dual of the above statement holds.

(e) Two maps $f, g : M \to N$ are equalized by an element of S if and only if they are coequalized by an element of S.

Example 1.18. Let $\mathscr{A} = R$-**Mod** for R a commutative ring with 1. Put

$$\mathbf{Ch}(R) := \mathbf{Ch}(R\text{-}\mathbf{Mod})$$

and consider the collection of quasi-isomorphisms (weak-equivalences) W. Then $W^{-1}\mathbf{Ch}(R)$ is a localization where two chain complexes $C_\bullet, D_\bullet$ are considered isomorphic if there exists a quasi-isomorphism $E_\bullet \to D_\bullet$ and an isomorphism $E_\bullet \to C_\bullet$. In particular, we have a diagram of the form:

$$C_\bullet \overset{\simeq}{\leftarrow} E_\bullet \overset{qis}{\to} D_\bullet$$

Theorem 1.19. *Let $\mathscr{A}$ be an abelian category and S a localizing class of morphisms. Then $S^{-1}\mathscr{A}$ is again abelian and the localization functor $Q : \mathscr{A} \to S^{-1}\mathscr{A}$ is exact.*

Proof. See [Mil] $\square$

In particular, this theorem tells us that localizing classes are the correct classes of morphisms to localize by.

What we would like to naively say is that $W^{-1}\mathbf{Ch}(R)$ is the derived category of R-modules. However, we want to get ride of one other type of equivalence which will not change (co)homology: chain homotopies.

Definition 1.20. Let $f, g : C_\bullet \to D_\bullet$ be chain maps. Then we say that f is chain homotopic to g (denoted $f \simeq g$) if there exists a chain map $P : C_\bullet \to D_\bullet[1]$ such that

$$f - g = \delta_D P + P\delta_C$$

In diagrams:

$$\begin{tikzcd} \ldots \arrow[r] & C_{i+1} \arrow[r, "\partial_C"] \arrow[d, "f_{i+1}, g_{i+1}"'] & C_i \arrow[r, "\partial_C"] \arrow[d, "f_i, g_i"] \arrow[dl, "P"'] & C_{i-1} \arrow[r, "\partial_C"] \arrow[d, "f_{i-1}, g_{i-1}"] \arrow[dl, "P"'] & \ldots \\ \ldots \arrow[r] & D_{i+1} \arrow[r, "\partial_D"'] & D_i \arrow[r, "\partial_D"'] & D_{i-1} \arrow[r, "\partial_D"'] & \ldots \end{tikzcd}$$

Lemma 1.21. *If $f \simeq g$ then $f_* = g_*$ on homology.*

Thus, chain homotopy classes of maps constitute collections of maps which behave identically on homology. For this reason, we only want a single representative from each of these classes.

Definition 1.22. Let $\mathscr{A}$ be an abelian category. The **homotopy category** of $\mathscr{A}$, denoted $K(\mathscr{A})$, is the category whose objects are chain complexes in $\mathscr{A}$, but whose morphisms are given by

$$\mathrm{Hom}_{K(\mathscr{A})}(C_\bullet, D_\bullet) = \mathrm{Hom}_{\mathbf{Ch}(\mathscr{A})}(C_\bullet, D_\bullet)/(\text{chain homotopy})$$

Now, we can formally define the Derived category associated to the abelian category $\mathscr{A}$.

Definition 1.23. The (unbounded) **Derived category** of $\mathscr{A}$, denoted $\mathbf{D}(\mathscr{A})$ is precisely defined as

$$\mathbf{D}(\mathscr{A}) := W^{-1}K(\mathscr{A})$$

where W consists of all quasi-isomorphisms in $\mathbf{Ch}(\mathscr{A})$. There are a few distinguished subcategories of $\mathbf{D}(\mathscr{A})$ denoted collectively as $\mathbf{D}^*(\mathscr{A})$ where $* \in \{b, +, -\}$ meaning bounded complexes, bounded below complexes, and bounded above complexes.

Remark 1.24. For some more motivation on why such a category holds any importance is as follows: this is the correct setting for homological algebra. The next theorem will bring a bit more light to this, but the moral idea here is that given a functor $F : \mathscr{A} \to \mathscr{B}$ which is left exact, we construct the derived functors R^iF which are the cohomology of F is some sense. However, in the act of taking quotients, we lose some information and thus want to deal directly with the complex given by $FI^\bullet$ for instance. To do this, we extend F to its *total* derived functor $\mathbf{R}F : \mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{B})$. As it turns out,

$$R^iF(M) = H^i(\mathbf{R}F(M))$$

The benefit however is that now we lose no information in passing to $\mathbf{R}F(C_\bullet)$. The "moral" part of this story is that $\mathbf{R}$ does not always have to exist. If $\mathscr{A}$ has enough injectives or projectives, then the total right or left derived functors exist. In this setting however, the story simplifies as

$$\mathbf{R}F(A) = FI^\bullet$$

where $A \to I^\bullet$ is an injective resolution (resp. projective resolution and $\mathbf{L}$).

Theorem 1.25. *$\mathbf{D}(\mathscr{A})$ is additive.*

Proof of 1.25. $\mathbf{D}^*(\mathscr{A})$ is the localization of the additive category $K^*(\mathscr{A})$ via quasi-isomorphisms. It suffices to show that the collection of quasi-isomorphisms, W, forms a localizing class. This however is straight forward and follows from the definitions. Hence, $\mathbf{D}(\mathscr{A})$ is additive. □

Now we give a characterization of Hom for the derived category on chain complexes concentrated in a single degree for modules.

Proposition 1.26. *Let $A, B \in R$-**Mod**. Then*

$$\mathrm{Hom}_{\mathbf{D}(R\text{-}\mathbf{Mod})}(A, B[i]) = \mathrm{Ext}^i_R(A, B)$$

Proof. See [Har66]. □

Example 1.27. Let $\mathscr{A} = \mathbf{Vect}_{\mathbb{C}}^{<\infty}$. We claim that $\mathbf{D}(\mathscr{A})$ is naturally equivalent to $\mathbb{Z}$-$\mathbf{GrVect}_{\mathbb{C}}$. To see this, first notice that given any chain complex of vector spaces, we can construct this complex (by choosing bases) from a direct sum of the chain complexes:

$$... \to 0 \to \mathbb{C} \to 0 \to ...$$

$$... \to 0 \to \mathbb{C} \to \mathbb{C} \to 0 \to ...$$

inductively. The second sequence is quasi-isomorphic to 0 and the first has homology k in degree 0 only. Thus every chain complex $V_\bullet$ is quasi-isomorphic to $\bigoplus_{i\in\mathbb{Z}} V_i[i]$ where V_i is some vector space. Now, by the above proposition $\mathrm{Hom}(k, k[j]) = \mathrm{Ext}^j_k(k,k) = 0$ if $j \neq 0$. Whence,

$$\mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(V_\bullet, W_\bullet) = \bigoplus_{\mathbb{Z}} \mathrm{Hom}_k(V_i, W_i)$$

which is exactly the morphisms in $\mathbb{Z}$-$\mathbf{GrVect}_{\mathbb{C}}$. Hence, the equivalence of categories is precisely send $V_\bullet$ to $\bigoplus_{i\in\mathbb{Z}} V_i$.

2 The Singularity Category

We now proceed with the more commutative algebra portion of the notes. For the remainder of the text, set $\mathscr{A} := R$-$\mathbf{Mod}$ for a local ring $(R, \mathfrak{m}, k)$. As we have seen, $\mathbf{D}^*(\mathscr{A})$ is an abelian category for $* \in \{b, +, -\}$.

We will construct a thick (see definition below) subcategory of $\mathbf{D}^b(\mathscr{A})$ and a quotient by this category will be the singularity category we desire. In combination with the theorem of Auslander-Buchsbaum-Serre, we will show that this category measures regularity of the ring R.

Definition 2.1. Let $\mathscr{B}$ be a full subcategory of $\mathscr{A}$. Then $\mathscr{B}$ is **thick** if for all exact sequences

$$0 \to M' \to M \to M'' \to 0$$

$M \in \mathscr{B}$ if and only if $M', M'' \in \mathscr{B}$.

Lemma 2.2. *Let $\mathscr{B}$ be a thick subcategory of $\mathscr{A}$. Let S be the set of morphisms whose kernel and cokernel are objects in $\mathscr{B}$. Then S is a localizing class and $\mathscr{A}/\mathscr{B} := S^{-1}\mathscr{A}$ is an abelian category.*

Proof. See [Mil] □

Corollary 2.3. *Every element of $\mathscr{B}$ is isomorphic to 0 in $\mathscr{A}/\mathscr{B}$.*

Proof. Consider the unique morphism $M \to 0$. Then $\ker = M$ and $\mathrm{coker} = 0$. □

Now that we have this language, we may define a certain thick subcategory of $\mathbf{D}^b(\mathscr{A})$.

Definition 2.4. A complex $C_\bullet$ in $Ch(\mathscr{A})$ is **perfect** if $C_\bullet$ is quasi-isomorphic to a bounded complex of finitely generated projective modules. Note that if $Q : K^b(\mathscr{A}_{fg}) \to \mathbf{D}^b(\mathscr{A}_{fg})$ is the localization functor, then $Q(C_\bullet)$ is isomorphic to this complex in the derived category. Set $\mathbf{Perf}(\mathscr{A})$ to be the subcategory of $\mathbf{D}^b(\mathscr{A}_{fg})$ consisting of perfect complexes with the natural morphisms.

Lemma 2.5. ***Perf***$(\mathscr{A})$ *is a thick subcategory of* $\mathbf{D}^b(\mathscr{A}_{fg})$.

Definition 2.6. The **singularity category** of $\mathscr{A}$ is the category

$$\mathbf{D}_{sg}(\mathscr{A}) := \frac{\mathbf{D}^b(\mathscr{A}_{fg})}{\mathbf{Perf}(\mathscr{A})}$$

Theorem 2.7. *$\mathbf{D}_{sg}(\mathscr{A}) \simeq 0$ if and only if R is regular.*

Proof. ($\Leftarrow$) Assume R is regular. Then by Auslander-Buchsbaum-Serre, every finitely generated R-module has finite projective dimension. In particular, every module is quasi-isomorphic to a perfect complex. Therefore, any complex in $\mathbf{D}^b(\mathscr{A}_{fg})$ is quasi-isomorphic to a perfect complex. As quasi-isomorphisms are isomorphisms in the derived category, we have that $\mathbf{D}_{sg}(\mathscr{A}) \simeq 0$ as every object is uniquely isomorphic to every other object.

($\Rightarrow$) Assume that $\mathbf{D}_{sg} \simeq 0$. In particular, every finitely generated module is perfect, hence has finite projective dimension. By ABS, R is regular. □

Now let S be a multiplicative set in R. Consider the canonical morphism $R \to S^{-1}R$.

Corollary 2.8. *If R is regular then $S^{-1}R$ is regular.*

Proof. We claim there exists an essentially surjective functor $\mathbf{D}_{sg}(R) \to \mathbf{D}_{sg}(S^{-1}R)$. To construct this, consider the functor $M \mapsto M \otimes_R S^{-1}R$ from R-**Mod** $\to$ $S^{-1}R$-**Mod** This is essentially surjective as there is an equivalence of categories

$$S^{-1}R\text{-}\mathbf{Mod} \simeq R\text{-}\mathbf{Mod}_S$$

where R-$\mathbf{Mod}_S$ is the category of R-modules where S acts by automorphisms. This map descends to the subcategories of finitely generated modules and thus their derived categories. Whence, defining an essentially surjective functor on singularity categories. Now, by the previous theorem R is regular if and only if $\mathbf{D}_{sg}(R) \simeq 0$. As the functor given is essentially surjective, $\mathbf{D}_{sg}(S^{-1}R) \simeq 0$. This completes the proof. □

Example 2.9. We will give two examples showing what happens for regular and non-regular rings.

(a) Consider $R = k[x]/(x^2)$ for k a field of characteristic 0. We claim this ring is not regular. To show this, we will exhibit k as quasi-isomorphic to an infinite resolution. Namely, notice that $R/(x) = k$. So, consider the complex

$$C_\bullet = \ldots \to R \xrightarrow{x} R \xrightarrow{x} R\ to 0$$

Then $H_i(C_\bullet) = 0$ for all $i > 0$ and $H_0(C_\bullet) = k$. In particular, k does not have finite projective dimension.

(b) Consider $k[x]$ and consider the maximal ideal $\mathfrak{m} = (x)$. Then set $R = k[x]_\mathfrak{m}$. This is regular as $\dim R = 1$ and $\mathfrak{m}R$ is principal in R. Now, a resolution of $k = R/\mathfrak{m}R$ is given by the Koszul complex:

$$0 \to R \xrightarrow{x} R \to 0$$

As R is a domain, and x is a regular element, this is a free resolution of k. In particular $\operatorname{projdim} k < \infty$.

References

[Har66] Robin Hartshorne. *Residues and Duality*, volume 20 of *Lecture Notes in Mathematics*. Springer-Verlag, 1966.

[Mil] Dragan Miličić. Lectures on derived categories. Unpublished notes.