Instructions:

- Answer each question in the space provided. Scratch paper will be provided and should be attached to your test when turned in, but your final answer should be presented as neatly as possible in the space provided.
- No calculators are allowed.
- You may refer to a note card no larger than 3 x 5 inches.
- Answers should be justified as appropriate.

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1. (10 points) Let

\[ A = \{1, 2, 3\} \]
\[ B = \{3, 4, 5\} \]
\[ C = \{1, 4\} \]

List all elements of the following sets:

(a) \( A \cup (B \cap C) \)
\( B \cap C = \{4\} \)
\( A \cup \{4\} = \{1, 2, 3, 4\} \)

(b) \((A - C) \cup B\)
\( A - C = \{2, 3\} \)
\( 1, 2, 3 \cup \{3, 4, 5\} = \{2, 3, 4, 5\} \)

(c) \( A - (C \cup B) \)
\( C \cup B = \{1, 3, 4, 5\} \)
\( A - \{1, 3, 4, 5\} = \{2\} \)

(d) \( A \times C \)
\( \{ (1,1), (1,4), (2,1), (2,4), (3,1), (3,4) \} \)

(e) \( P(C) \) (the power set of \( C \))
\( \{ \emptyset, \{1\}, \{4\}, \{1,4\} \} \)
2. (10 points) Consider the function \( f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \) given by \( f(x, y) = x - y \).

(a) Show that \( f \) is not one-to-one.

\[
\text{Since } f(2, 5) = f(1, 4) = -3, \text{ } f \text{ is not one-to-one.}
\]

(b) Show that \( f \) is onto.

\[
\text{Let } n \in \mathbb{Z}.
\]

\[
\text{Since } f(n, 0) = n, \text{ } f \text{ is onto.}
\]
3. (10 points) Consider the relation \( \sim \) on the set \( \mathbb{Z} \times \mathbb{Z} \) given by \( (a, b) \sim (c, d) \) if \( a \equiv c \pmod{2} \) and \( b \equiv d \pmod{3} \).

(a) Show that \( \sim \) is reflexive.

Let \( (a, b) \in \mathbb{Z} \times \mathbb{Z} \).

Since \( a \equiv a \pmod{2} \) and \( b \equiv b \pmod{3} \), \( (a, b) \sim (a, b) \)

and \( \sim \) is reflexive.

(b) Show that \( \sim \) is symmetric.

Let \( (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \) such that \( (a, b) \sim (c, d) \).

Then \( a \equiv c \pmod{2} \), which implies \( c \equiv a \pmod{2} \).

Similarly, \( b \equiv d \pmod{3} \) means \( d \equiv b \pmod{3} \).

So \( (c, d) \sim (a, b) \) and \( \sim \) is symmetric.

(c) Show that \( \sim \) is not antisymmetric.

Since \( (0, 0) \sim (7, 3) \) and \( (2, 3) \sim (0, 0) \) but \( (0, 0) \neq (7, 3) \),

\( \sim \) is not antisymmetric.

(d) Show that \( \sim \) is transitive.

Let \( (a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z} \) such that \( (a, b) \sim (c, d) \)

and \( (c, d) \sim (e, f) \).

Since \( a \equiv c \pmod{2} \) and \( c \equiv e \pmod{2} \), \( a \equiv e \pmod{2} \).

Since \( b \equiv d \pmod{3} \) and \( d \equiv f \pmod{3} \), \( b \equiv f \pmod{3} \).

So \( (a, b) \sim (e, f) \) and \( \sim \) is transitive.
4. (10 points) Consider the relation \( \sim \) on the set \( \mathbb{Z} \times \mathbb{Z} \) given by 
\((a, b) \sim (c, d)\) if \( a \equiv c \pmod{2} \) and \( b \equiv d \pmod{3} \) as in the previous problem. Problem 3 shows that this is an equivalence relation.

(a) List four elements of \( \mathbb{Z} \times \mathbb{Z} \) in the same equivalence class as \((0, 0)\).

\[
\begin{align*}
(0, 0) \\
(2, 3) \\
(-2, -3) \\
(0, 0) \\
(4, 10) \\
\vdots
\end{align*}
\]

(b) List all equivalence classes of the relation.

\[
\begin{align*}
[(0, 0)] \\
[(0, 1)] \\
[(0, 2)] \\
[(1, 0)] \\
[(1, 1)] \\
[(1, 2)]
\end{align*}
\]
5. (10 points) Prove or disprove each of the following statements:

(a) If \(a|c\) and \(b|c\), then \((ab)|c\).

\[
\text{False: } 2|4, 4|4 \text{ but } 8 \nmid 4.
\]

(b) If \(x|y\) and \(x|z\), then \(x|yz\).

By the definition of divisors, \(\exists s, t \in \mathbb{Z} \text{ s.t. } y = sx \text{ and } z = tx\).

\[
yz = sx \cdot tx = (sx)tx = (st)x.
\]

Since \(stx\) is an integer, \(x|yz\).
6. (5 points) Find the greatest common divisor of 222 and 629.

\[
\begin{align*}
629 & = 2 \cdot 222 + 185 \\
222 & = 185 + 37 \\
185 & = 5 \cdot 37 + 0
\end{align*}
\]

So \( \gcd(629, 222) = \gcd(37, 0) = 37 \)

7. (5 points) Find integers \( s \) and \( t \) such that \( 17s + 12t = 1 \).

\[
\begin{align*}
17 & = 1 \cdot 12 + 5 \\
12 & = 2 \cdot 5 + 2 \\
5 & = 2 \cdot 2 + 1 \\
1 & = 5 - 2 \cdot 2 \\
1 & = 5 - 2(12 - 2 \cdot 5) \\
1 & = 5 \cdot 5 - 2 \cdot 12 \\
1 & = 5(17 - 12) - 2 \cdot 12 \\
1 & = 5(17) - 7 \cdot 12 \\
s & = 5, \; t = -7
\end{align*}
\]
8. (5 points (bonus)) Let \(a, n \in \mathbb{Z}\) and \(n > 0\). Show that the following are equivalent:

(i) \(a \equiv 0 \pmod{n}\)

(ii) \(a \mod n = 0\)

(iii) \(n|a\)

(i) \(\rightarrow\) (ii):

If \(a \equiv 0 \pmod{n}\), then \(n|a - 0\) and \(n|a\). Therefore, there is some integer \(q\) such that \(a = nq + 0\).

By definition of mod, \(a \mod n = 0\).

(ii) \(\rightarrow\) (iii)

If \(a \mod n = 0\), there is some integer \(q\) such that \(a = qn + 0\). Since \(a = qn\), \(n|a\).

(iii) \(\rightarrow\) (i)

If \(n|a\), then \(n|\left(a - 0\right)\). Therefore, \(a \equiv 0 \pmod{n}\).