Let \( A \) be the set of numbers divisible by 6 and \( B \) the set of numbers divisible by 3.

* Show that \( A \subseteq B \).

Let \( a \in A \). Since \( 6 | a \), \( \exists k \in \mathbb{Z} \) such that \( a = 6k \).
Then \( a = 3(2k) \), so \( 3 | a \) and \( a \in B \). \( \Box \)

* Show that \( A \nsubseteq B \).

\( 3 \in B \), but \( 3 \notin A \). Therefore \( A \nsubseteq B \).

\[
\begin{align*}
\text{Let } & A = \{ n \in \mathbb{Z} \mid n \text{ is odd} \} \\
& B = \{ n \in \mathbb{Z} \mid n \mod 4 = 1 \} \\
& C = \{ n \in \mathbb{Z} \mid n \mod 4 = 3 \} \\
\text{Show that } & A = B \cup C.
\end{align*}
\]

Let \( a \in A \).
Then there is some \( k \in \mathbb{Z} \) st. \( a = 2k+1 \)

Case 1: \( k \) is even.
Then there is some \( \ell \in \mathbb{Z} \) st. \( k = 2\ell \).
\( a = 2(2\ell) + 1 = 4\ell + 1 \)
So \( a \mod 4 = 1 \) and \( a \in B \)

Case 2: \( k \) is odd.
Then there is some \( \ell \in \mathbb{Z} \) st. \( k = 2\ell + 1 \).
\( a = 2(2\ell + 1) + 1 = 4\ell + 3 \)
So \( a \mod 4 = 3 \) and \( a \in C \).

So \( a \in B \cup C \) and \( A \subseteq B \cup C \).

(continued on next page.)
Let $x \in B \cup C$.
Then $x \in B$ or $x \in C$.

Case 1: $x \in B$; that is, $x \mod 4 = 1$.
Then there is some $n \in \mathbb{Z}$ such that $x = 4n + 1$.
Since $x = 2(2n + 1) + 1$, $x$ is odd and $x \in A$.

Case 2: $x \in C$; that is, $x \mod 4 = 3$.
Then there is some $n \in \mathbb{Z}$ such that $x = 4n + 3$.
Since $x = 2(2n + 1) + 1$, $x$ is odd and $x \in A$.

Therefore $B \cup C \subseteq A$.

Thus $A = B \cup C$.

Let $f: \mathbb{R} \rightarrow \mathbb{Z}$ be the function that maps each real number $x$ to the largest integer $y$ such that $y \leq x$.

* $f$ is onto:
  Let $z \in \mathbb{Z}$. Then $f(z) = z$, so $f$ is onto.

* $f$ is not one-to-one:
  
  $f(3.1) = f(3.2) = 3$

Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(n) = 3n + 2$.

* $f$ is not onto:
  There is no integer $n$ such that $3n + 2 = 0$.

* $f$ is injective:
  Suppose $f(m) = f(n)$.
  Then $3m + 2 = 3n + 2$.
  
  $3m = 3n$
  $m = n$. 

(2)
Let $\sim$ denote the relation on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) \sim (c, d)$ if there is some $k \in \mathbb{Q}$ such that $a = kc$ and $b = kd$. Show that $\sim$ is an equivalence relation.

Let $(a, b) \in \mathbb{Z} \times \mathbb{Z}$
Since $a = 1 \cdot a$ and $b = 1 \cdot b$, $(a, b) \sim (a, b)$.
Thus $\sim$ is reflexive.

Let $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$ with $(a, b) \sim (c, d)$.
Then $\exists k \in \mathbb{Q}$ s.t. $a = kc$ and $b = kd$.
Thus $c = \frac{1}{k} a$ and $d = \frac{1}{k} b$ and $(c, d) \sim (a, b)$.
Thus $\sim$ is symmetric.

Let $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z}$ with $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
Then there are integers $k, l$ such that $a = kc$, $b = kd$, $c = le$, $d = lf$.
Denote $a = k(ke) = (ke)e$
$b = k(lf) = (kf)f$

So $(a, b) \sim (e, f)$ and $\sim$ is transitive.

Thus $\sim$ is an equivalence relation.
By the fundamental theorem of arithmetic, any integer \( a \) with \( a \geq 2 \) can be written uniquely in the form \( a = 2^n k \) for an odd number \( k \). Define a relation \( \preceq \) on the set \( A = \{ z \in \mathbb{Z} \mid z \geq 2 \} \) by \( 2^n k \preceq 2^m l \) if \( k \) and \( l \) are odd numbers and \( n \leq m \).

Show that \( \preceq \) is a partial order.

\( \preceq \) is reflexive: Let \( a = 2^n k \) be an integer. Since \( n \leq n \), \( 2^n k \preceq 2^n k \).

So \( \preceq \) is reflexive.

\( \preceq \) is antisymmetric:

This is false.

e.g. \( 6 \preceq 10 \) and \( 10 \preceq 6 \).

Oops.

\( \preceq \) is transitive:

Let \( 2^n k, 2^m l, 2^a b \) be \( A \) with \( 2^n k \preceq 2^m l \) and \( 2^m l \preceq 2^a b \). Then \( n \leq m \) and \( m \leq a \). Since \( \preceq \) is transitive, \( n \leq a \) and \( 2^n k \preceq 2^a b \).

So \( \preceq \) is a reflexive and transitive relation, but not a partial order.