Some examples of well-written inductive proofs

Theorem: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for every positive integer $n$.

Proof: Base case: $\frac{1(1+1)}{2} = 1$

Inductive case: Suppose $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Then $1 + 2 + \cdots + (k+1) = \frac{k(k+1)}{2} + (k+1)$

$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{k(k+1) + 2(k+1)}{2}$

$= \frac{(k+2)(k+1)}{2}$

$= \frac{(k+1)((k+1)+1)}{2}$

Q.E.D.
Theorem: $2^n < n!$ for every integer $n$ with $n > 4$.

Proof: Base case: $2^4 = 16 \quad 4! = 24$

So $2^4 < 4!$

Inductive case: Suppose $2^k < k!$ for some $k > 4$.

Since $2 < k+1$, we have $2^k (k!) < (k+1) \cdot k!$

and $2 \cdot k! < (k+1)k!$

Since $2^k < k!$, we have $2 \cdot 2^k < 2 \cdot k!$

and $2^{k+1} < 2 \cdot k!$

Since $<$ is transitive, we have $2^{k+1} < (k+1)!$

Q.E.D.
Suppose n people have a pie fight: each person simultaneously (and accurately) throws a pie at the nearest person (assume there are no ties.) Anyone not hit with a pie is considered a survivor.

Theorem: In a pie fight with an odd number of participants, there is always at least one survivor.

Proof: Let P(n) be the statement "In a pie fight with 2n+1 people, there is always one survivor.

Base case: Consider a pie fight between 3 people. Let A and B be the closest pair. Then A and B throw pies at each other and the third person is a survivor. Therefore P(1).

Inductive case: Suppose there is always a survivor in a pie fight between 2n+1 people. Consider a pie fight among 2n+3 people. Let A and B be the closest pair, so that A and B throw pies at each other.

If someone else throws a pie at either A or B, the remaining 2n+1 people will be hit with at most 2n pies, leaving at least one survivor.

If no one else throws a pie at A or B, the remaining 2n+1 people have a pie fight among themselves, which leaves at least one survivor by the inductive hypothesis.

Thus, any case leaves at least one survivor.

QED
Show that any natural number $n > 1$ can be written as a product of primes.
(Uses strong induction)

Proof:

Base case: Since 2 is prime, it is a product of one prime.

Inductive case: Suppose every integer $i$ with $2 \leq i \leq k$ can be written as a product of primes. Consider $k+1$.

If $k+1$ is a prime, then $k+1$ is a product of one prime.

If $k+1$ is composite, it has a prime factor $p < k$ so that $k+1 = p \cdot \frac{k+1}{p}$. Then $\frac{k+1}{p}$ is an integer, $1 < \frac{k+1}{p} < k+1$, so $\frac{k+1}{p}$ may be written as a product of primes $q_1, q_2, ..., q_e$.

Since $k+1 = p q_1 \cdots q_e$, $k+1$ is a product of primes.

QED
A post office sells only 4¢ and 5¢ stamps. Show that any amount of postage 12¢ or greater can be made from these stamps.

Proof 1 (standard induction):

Base case: Since 12¢ = 3·4¢, 12¢ can be made from 4¢ and 5¢ stamps.

Inductive case: Suppose $k > 12$ and it is possible to make $k¢$ from 4¢ and 5¢ stamps. We will show that it is possible to make $(k+1)¢$ from 4¢ and 5¢ stamps.

Case 1: We can make $k¢$ using at least one 4¢ stamp. In this case, replacing the 4¢ stamp with a 5¢ stamp gives us $(k+1)¢$ of postage.

Case 2: We can make $k¢$ using only 5¢ stamps. Since $5k$ and $k > 12$, we conclude that $k > 15$ and therefore at least three 5¢ stamps are required. In this case, replacing two three 5¢ stamps with four 4¢ stamps gives us $(k+1)¢$ of postage.

\[\text{QED}\]
Proof 2: Strong Induction.

Let $P(n)$ be the statement "n¢ of postage can be made from 4¢ and 5¢ stamps."

Base case:

$P(12)$ since $12¢ = 3\times 4¢$

$P(13)$ since $13¢ = 2\times 4¢ + 1\times 5¢$

$P(14)$ since $14¢ = 4¢ + 2\times 5¢$

$P(15)$ since $15¢ = 3\times 5¢$

Inductive case:

Let $k > 15$ and suppose for any $i$ with $12 \leq i \leq k$, $P(i)$ is true.

Since $12 \leq k-3 \leq k$, we can make $(k-3)¢$ from 4¢ and 5¢ stamps.

Since $(k-3)¢ + 4¢ = (k+1)¢$, we can add one 4¢ stamp to our $(k-3)¢$ to get $(k+1)¢$.

QED