Sample questions for Quiz 3.

(1) Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \), given by
\[
f(x, y) = \begin{cases} 
\frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\
0, & (x, y) \neq (0, 0)
\end{cases}
\]
Show that both \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) exist everywhere, but the function \( f \) is not differentiable at \((0, 0)\).

(2) Find the differential matrix for the function \( G : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R} \) defined by
\[
G(x, y) = (y \ln x, xe^{xy}, \sin(xy)).
\]
Then find the best affine approximation to \( G \) at the point \((1, \frac{\pi}{2})\).

(3) Assume the function \( F : \mathbb{R}^p \to \mathbb{R} \) is differentiable everywhere, and that its gradient \( dF \) is a constant vector. Show that \( F \) is an affine function.

(4) Find the differential of the real values function \( f(x, y, z) = xy^2 \cos xz \). Then find the best affine approximation to \( f \) at the point \((1, 1, \frac{\pi}{2})\).

(5) From the definitions, prove that if a function \( F : U \to \mathbb{R}^q, U \subset \mathbb{R}^p \) open, is differentiable at \( a \in \mathbb{R}^p \), then \( F \) is continuous at \( a \).

(6) If \( f : \mathbb{R} \to \mathbb{R} \) is differentiable, and \( g(x, y) = f(xy) \), show that
\[
x \frac{\partial g}{\partial x} - y \frac{\partial g}{\partial y} = 0.
\]

(7) If \((x, y)\) are the Cartesian coordinates in the plane, the polar coordinates are
\[
x = r \cos \theta, \quad y = r \sin \theta.
\]
(a) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a differentiable function in \( x, y \). Find formulas for the partial derivatives of \( f \) with respect to \( r \) and \( \theta \) in terms of the partial derivatives with respect to \( x \) and \( y \).

(b) An important operator in mathematics is the Laplacian \( \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \). Prove that in polar coordinates, the Laplacian is given by
\[
\Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}.
\]

(8) For the curve \( \gamma(t) = (\cos t, \sin 2t) \), find a parametric equation of the tangent line at \((0, 0)\) if the domain of \( \gamma(t) \) is \( \{ t : \pi < t < 2\pi \} \).

(9) Find an equation for the tangent plane to the surface \( x^2 + y^2 - z^2 = 1 \) at each point \( (a, b, c) \) on the surface.

(10) Find the degree \( n = 2 \) Taylor’s formula for \( f(x, y) = \ln(x + y + 1) \) at the point \((0, 0)\).

(11) Find all points of local maximum and local minimum and all saddle points for \( f(x, y) = y^3 + y^2 + x^2 - 2xy - 3y \).

(12) Find where the function \( f(x, y, z) = 2xy - z \) attains its maximum and minimum values on the sphere \( S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \).

(13) Consider \( F : \mathbb{R}^2 \to \mathbb{R}^2, F(x, y) = (e^x \cos y, e^x \sin y) \).
(a) Does \( F \) have an inverse?
(b) Does \( F \) have a smooth local inverse function near every point \((x, y)\)?
(14) Consider \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, F(r, \theta) = (r \cos \theta, r \sin \theta) \). Does a smooth local inverse function exist near \((1, 2\pi)\)? If it does, find one explicitly.

(15) Consider the function \( F : \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}^2, F(x, y) = (x^2, \frac{y}{x}) \).

(a) Find all the points near which \( F \) has a smooth local inverse.

(b) If \( F \) is restricted to \( V = \{(x, y) : x > 0, y > 0\} \), find a smooth inverse for \( F \). What is the domain of \( F^{-1} \)?

(c) As in (b), find the differential of \( F^{-1} \) directly, and also by applying the Inverse Function Theorem. Compare the results.

(16) Exercises 4 and 5 in 9.7.