1. (5 points) Express \( \ln(12) \) in terms of \( \ln(2) \) and \( \ln(3) \).

Solution:

\[
\ln(12) = \ln(2^2 \cdot 3) = \ln(2^2) + \ln(3) = 2 \cdot \ln(2) + \ln(3)
\]

2. (5 points) If \( g(x) \) is pictured in Figure 1 and

\[
f(x) = \int_0^x g(t) \, dt \quad ; \quad x \geq 0,
\]

is \( f(x) \) invertible?

Solution:

Yes. \( f(x) \) is an accumulation function and \( g(x) \) is strictly positive, so \( f(x) \) is strictly increasing. Monotonic functions have inverses.

3. (5 points) Suppose \( \frac{dy}{dx} = f(x) \), \( f(0) = 1 \), and \( f(x) \) is decreasing. Suppose you use Euler’s Method to approximate \( y(1) \). Will your estimate be an overestimate or an underestimate?

Solution:

If \( f(x) \) is decreasing then \( y \) is concave down, so \( y \) curves beneath its tangent lines. Euler’s Method will give an overestimate.

4. (5 points) \( f(x) \) is shown in Figure 2. Does \( f(x) \) have an inverse on its entire domain? Sketch a graph of \( f^{-1}(x) \), restricting the domain if necessary.
Figure 2: \( f(x) \)

**Solution:**

\( f(x) \) has an inverse since it is strictly increasing. To sketch a graph of the inverse, reflect diagonally.

5. **(5 points)** Suppose \( T(t) = 100 - (100 - 50)e^{-t} \) describes the temperature of an object with respect to time. Is the object getting hotter or colder? What temperature is it
approaching?

Solution:

The object is getting hotter since \( T'(t) = 50e^{-t} \) is positive. \( T(t) \) is approaching 100 since \( e^{-t} \) goes to 0 as \( t \) increases.

6. (5 points) What is the derivative of \( e^{\cosh x} \)?

Solution:

Let \( w = \cosh(x) \).

\[
\frac{d}{dx} (e^{\cosh(x)}) = \frac{d}{dx} (e^w) \\
= \frac{d(e^w)}{dw} \cdot \frac{dw}{dx} \\
= e^w \cdot \frac{dw}{dx} \\
= e^{\cosh(x)} \cdot \sinh(x)
\]

7. (5 points) \( \frac{dy}{dx} = ky, \ y(0) = 2 \) and \( y'(0) = 4 \). Find \( y(x) \).

Solution:

\[
4 = y'(0) = \frac{dy}{dx}(0) = k \cdot y(0) = 2k
\]

So \( k = 2 \).

\[
\frac{dy}{dx} = 2y \\
\frac{1}{y} dy = 2 dx \\
\int \frac{1}{y} dy = \int 2 dx \\
\ln(y) = 2x + C \\
y = e^{-2x+C} = C_2 e^{2x}
\]

\[
y(0) = 2 = C_2
\]

So

\[
y = 2e^{2x}
\]
8. (5 points) Is $2 \sin(2x)$ a solution to $\frac{d^2 y}{dx^2} - 4y = 0$?

Solution:

No.

$$\frac{d^2}{dx^2}(2 \sin(2x)) - 4(2 \sin(2x)) = -8 \sin(2x) - 8 \sin(2x) = -16 \sin(2x)$$

This function is not identically zero, so $2 \sin(2x)$ is not a solution.

9. (5 points) Calculate

$$\int_0^2 \frac{2}{4 + x^2} dx$$

Solution:

$$\int_0^2 \frac{2}{4 + x^2} dx = \int_0^2 \frac{2}{4(1 + \frac{x^2}{4})} dx$$

substitute $w = \frac{x}{2}$

$$= \int_0^1 \frac{2}{4(1 + w^2)} 2dw$$

$$= \int_0^1 \frac{1}{1 + w^2} dw$$

$$= \arctan(w) \bigg|_{w=0}^{1}$$

$$= \arctan(1) - \arctan(0)$$

$$= \frac{\pi}{4}$$

10. (5 points) $f(x) = x^2 - 3x + 1$. Restrict the domain of $f(x)$ so that it has an inverse, while keeping its range as large as possible. Then find $f^{-1}(x)$.

Solution:

Consider the derivative $f'(x) = 2x - 3$. This is negative for $x < \frac{3}{2}$ and positive for $x > \frac{3}{2}$. Thus, $x = \frac{3}{2}$ is the global minimum for $f(x)$, and if we restrict the domain to either $(-\infty, \frac{3}{2}]$ or $[\frac{3}{2}, \infty)$ then $f(x)$ will be strictly monotonic, hence, invertible.

We will choose to restrict the domain to $[\frac{3}{2}, \infty)$. The range of $f$ is $y \geq -\frac{5}{4}$.  
\[ f(x) = y = x^2 - 3x + 1 \]
\[ 0 = x^2 - 3x + (1 - y) \]
\[ x = \frac{3 \pm \sqrt{9 - 4(1 - y)}}{2} \quad \text{(by quadratic formula)} \]
\[ x = \frac{3 + \sqrt{9 - 4(1 - y)}}{2} \quad \text{(since } x \geq \frac{3}{2}) \]
\[ x = \frac{3 + \sqrt{1 + 4y}}{2} \]
\[ f^{-1}(x) = \frac{3 + \sqrt{1 + 4x}}{2}; \quad x \geq -\frac{5}{4} \]

11. **(15 points)** A batch of brownies cooks in a 350° oven. The temperature in the room is 70° and the temperature inside the refrigerator is 38°. The brownies are taken out of the oven and placed on the counter. After 5 minutes you try a brownie but burn yourself, they are still 210°. How much longer must you wait until the brownies reach 110° and are safe to eat? Newton’s law of cooling gives us the formula:

\[ T(t) = T_A + (T_0 - T_A)e^{-kt} \]

**Solution:**

\[ T_A = 70 \text{ and } T_0 = 350. \] We also know that

\[ T(5) = 210 = 70 + (350 - 70)e^{-5k} \]
\[ \frac{140}{280} = e^{-5k} \]
\[ k = -\frac{1}{5} \ln\left(\frac{140}{280}\right) = \frac{\ln(2)}{5} \]

Knowing \( k \) we can now solve for the time when the brownies will be 110°.

\[ T(t) = 110 = 70 + (350 - 70)e^{-kt} \]
\[ \frac{40}{280} = e^{-kt} \]
\[ t = -\frac{1}{k} \ln\left(\frac{1}{7}\right) = \frac{\ln(7)}{k} = \frac{5 \ln(7)}{\ln(2)} \approx 14 \]

So the brownies will be cool in another 9 minutes.
12. **(15 points)** A tank initially contains 50 gallons of brine, with 30 lbs of salt in solution. Fresh water runs into the tank at 3 gallons per minute, and the well stirred solution runs out at 2 gallons per minute. How long will it take until there are 25 lbs of salt in the tank?

**Solution:**

Let $A(t)$ be the amount of salt in pounds in the tank after $t$ minutes. Let $V(t)$ be the volume of water in the tank. $A(0) = 30$. $V(0) = 50$.

The volume in the tank is not constant, more water enters the tank than leaves.

$$\frac{dV}{dt} = \text{rate in} - \text{rate out}$$

$$= 3 \text{ gal/min} - 2 \text{ gal/min} = 1 \text{ gal/min}$$

$$V(t) = t + V_0 = t + 50$$

$$\frac{dA(t)}{dt} = \text{rate in} - \text{rate out}$$

$$= (3 \text{ gal/min})(0 \text{ lbs/gal}) - (2 \text{ gal/min}) \frac{A(t) \text{ lbs}}{V(t) \text{ gal}}$$

$$\frac{dA(t)}{dt} = -2 \frac{A(t)}{t + 50}$$

$$\frac{1}{A(t)} dA(t) = -\frac{2}{t + 50} dt$$

$$\int \frac{1}{A(t)} dA(t) = \int -\frac{2}{t + 50} dt$$

$$\ln(A(t)) = -2 \ln(t + 50) + C$$

$$A(t) = e^{-2 \ln(t+50)+C} = Be^{-2 \ln(t+50)}$$

$$A(t) = B(t + 50)^{-2}$$

$$A(0) = 30 = B \cdot 50^{-2}$$

$$B = 30 \cdot 50^2$$

$$A(t) = 30 \cdot 50^2 \cdot (t + 50)^{-2}$$
Now if \( A(t) = 25 \) then

\[
A(t) = 25 = 30 \cdot 50^2 \cdot (t + 50)^{-2}
\]

\[
(t + 50)^2 = 3000
\]

\[
t = -50 + \sqrt{3000} \approx 4.77 \text{ min}
\]

13. (20 points) In 2004 the population of the world was \( 6.4 \times 10^9 \) people and was increasing at a rate of \( 8.448 \times 10^7 \) people per year. Assume that the maximum population that the world can support is \( L = 1.728 \times 10^{10} \). Using the logistic model for population growth, determine the year in which the rate of population growth begins to slow.

\[
P(t) = \frac{LP_0}{P_0 + (L - P_0)e^{-Lht}}
\]

where \( \frac{dP}{dt}(t) = hP(t)(L - P(t)) \).

\[\text{Solution:}\]

\[
\frac{dP}{dt}(0) = hP_0(L - P_0)
\]

\[
h = \frac{\frac{dP}{dt}(0)}{P_0(L - P_0)}
\]

The rate of population growth will be greatest when the second derivative of population is zero.

\[
\frac{d^2P(t)}{dt^2} = hP(t)(-\frac{dP(t)}{dt}) + h \frac{dP(t)}{dt}(L - P(t))
\]

\[
= h(L - 2P(t)) \frac{dP(t)}{dt}
\]

So the population grows at the highest rate when \( P(t) = \frac{L}{2} \).
\[
\frac{L}{2} = P(t) = \frac{LP_0}{P_0 + (L - P_0)e^{-Lht}}
\]

\[
P_0 + (L - P_0)e^{-Lht} = 2P_0
\]

\[
e^{-Lht} = \frac{P_0}{L - P_0}
\]

\[
-Lht = \ln\left(\frac{P_0}{L - P_0}\right)
\]

\[
t = \frac{1}{Lh} \ln\left(\frac{L - P_0}{P_0}\right) \approx 25.3
\]

So the rate of population growth begins to slow in 2029.