Final exam notes for Math 3210

**Limits.** Let \( \{a_n\} \) be a sequence. Then

\[
\lim_{n \to \infty} a_n = a
\]

if for all \( \epsilon > 0 \) there exists an \( N \) such that if \( n > N \) then \( |a_n - a| < \epsilon \). If no such \( a \) exists then the sequence is **divergent**. The sequence \( a_n \) is **Cauchy** if for all \( \epsilon > 0 \) there exists an \( N > 0 \) such that if \( n, m > N \) then \( |a_n - a_m| \leq \epsilon \).

**Theorem 0.1** A sequence is convergent if and only if it is Cauchy.

**Theorem 0.2** Every bounded sequence of real numbers has a convergent subsequence.

**Theorem 0.3** Suppose \( a_n \to a \), \( b_n \to b \), \( c \) is a real number and \( k \) a natural number. Then

1. \( ca_n \to ca \);
2. \( a_n + b_n \to a + b \);
3. \( a_nb_n \to ab \);
4. \( a_n/b_n \to a/b \) if \( b \neq 0 \) and \( b_n \neq 0 \) for all \( n \);
5. \( a_n^k \to a^k \);
6. \( a_n^{1/k} \to a^{1/k} \) if \( a_n \geq 0 \) for all \( n \).

If \( A \) is a subset of \( \mathbb{R} \) the \( a = \sup A \) if \( a \geq x \) for all \( x \in A \) and \( a' \geq x \) for all \( x \in A \) then \( x \leq y \). We define \( \inf A \) be reversing the inequalities. If we allow \( +\infty \) and \( -\infty \) the \( \sup A \) and \( \inf A \) always exist.

Let \( \{a_n\} \) be a sequence and define \( i_n = \inf\{a_k : k \geq n\} \) and \( s_n = \sup\{a_k : k \geq n\} \). Then

\[
\lim \inf a_n = \lim i_n
\]

and

\[
\lim \sup a_n = \lim s_n.
\]

**Continuity.** Let \( f : D \to \mathbb{R} \) be a function defined on a domain \( D \subset \mathbb{R} \). Then

\[
\lim_{x \to a} f = b
\]

if for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if for all \( x \in D \) with \( 0 < |x - a| < \delta \) then \( |f(x) - b| < \epsilon \). The function \( f \) is **continuous** at \( a \) if

\[
\lim_{x \to a} f = f(a)
\]

There is a theorem similar Theorem 0.3 for limits of functions.

The function \( f \) is **uniformly continuous** if for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( x, y \in D \) and \( |x - y| < \delta \) then \( |f(x) - f(y)| < \epsilon \).
Theorem 0.4 Let \( f : [a, b] \to \mathbb{R} \) be continuous. Then there exists a \( c \) and \( d \) in \([a, b]\) such that \( f(x) \leq f(c) \) and \( f(x) \geq f(d) \) for all \( x \in [a, b] \).

Theorem 0.5 (Intermediate Value Theorem) Let \( f : [a, b] \to \mathbb{R} \) be continuous. If \( y \) is between \( f(a) \) and \( f(b) \) then there exists a \( x \in [a, b] \) such that \( f(c) = y \).

Theorem 0.6 Let \( f : [a, b] \to \mathbb{R} \) be continuous. Then \( f \) is uniformly continuous.

A sequence of functions \( f_n : D \to \mathbb{R} \) converges uniformly to \( f : D \to \mathbb{R} \) if for all \( \epsilon > 0 \) there exists an \( N > 0 \) such that if \( n > N \) then \( |f_n(x) - f(x)| < \epsilon \) for all \( x \in D \).

Theorem 0.7 Let \( f_n : D \to \mathbb{R} \) be continuous. If \( f_n \to f \) uniformly then \( f \) is continuous.

Derivatives. Define the derivative \( f'(a) \) of the function \( f \) at \( a \) by

\[
  f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

if it exists.

Differentiation rules (abbreviated):

1. \((f + g)'(a) = f'(a) + g'(a)\);
2. \((fg)(a) = f'(a)g(a) + f(a)g'(a)\);
3. \((f/g)(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}\);
4. \((f \circ g)'(a) = f'(g(a))g'(a)\)

Theorem 0.8 (Mean Value Theorem) Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists a \( c \in (a, b) \) such that

\[
  f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Theorem 0.9 (L'Hôpital's Rule) If \( f(x), g(x) \to 0 \) or \( f(x), g(x) \to \infty \) as \( x \to a \) then

\[
  \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.
\]
Integrals. Let $P = \{x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$ and for $k = 1, \ldots, n$ set

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} \text{ and } m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}.$$ 

We then define the upper and lower sums for $P$ by

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

and

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

We define the upper and lower integrals by

$$\int_{a}^{b} f(x)dx = \inf\{U(f, P) : P \text{ is a partition of } [0, 1]\}$$

and

$$\int_{a}^{b} f(x)dx = \sup\{L(f, P) : P \text{ is a partition of } [0, 1]\}.$$ 

Then $f$ is integrable if $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(x)dx$ and we write

$$\int_{a}^{b} f(x)dx = \int_{a}^{b} f(x)dx = \int_{a}^{b} f(x)dx.$$

**Theorem 0.10** $f$ is integrable $\iff$ for all $\varepsilon > 0$ there exist a partition $P$ such that $U(f, P) - L(f, P) < \varepsilon$ $\iff$ there exists partitions $P_n$ such that $U(f, P_n) - L(f, P_n) \to 0$.

**Properties of integrals (abbreviated):**

1. $\int c f = c \int f$ if $c \in \mathbb{R}$;
2. $\int f + \int g = \int f + g$;
3. $|\int f| \leq \int |f|$;
4. $\int_{a}^{b} f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(u)du$;
5. $\int_{a}^{b} f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_{a}^{b} f'(x)g(x)dx$

**Theorem 0.11** (Fundamental Theorems of Calculus)
1. \[ \int_a^b f'(x)\,dx = f(b) - f(a) \]

2. Define

\[ F(x) = \int_a^x f(t)\,dt. \]

If \( f \) is continuous at \( x \) then \( F'(x) = f(x) \).

**Series.** Let \( \{a_n\} \) be a sequence. Then the series \( \sum_{k=0}^{\infty} a_k \) converges if the sequence of partial sums \( s_n = \sum_{k=0}^{n} a_k \) converges. If \( \sum_{k=0}^{\infty} |a_k| \) converges then the series \( \sum_{k=0}^{\infty} a_k \) converges absolutely. If \( \sum_{k=0}^{\infty} |a_k| \) doesn’t converge but \( \sum_{k=0}^{\infty} a_k \) does then the series converges conditionally.

Tests for convergence and divergence:

1. If \( \sum_{k=0}^{\infty} a_n \) converges then \( a_n \to 0 \).
2. If \( a_n \geq b_n \) and \( \sum_{k=0}^{\infty} b_k \) converges then \( \sum_{k=0}^{\infty} a_k \) converges absolutely.
3. Let \( \{a_n\} \) be a sequence with \( 0 \leq a_{n+1} \leq a_n \) and let \( f : [0, \infty) \to \mathbb{R} \) be a non-increasing function such that \( f(n) = a_n \). Then \( \sum_{k=1}^{\infty} a_k \) converges \( \iff \int_1^{\infty} f(t)\,dt \) converges. If \( \sum_{k=1}^{\infty} a_k \) converges then

\[
\int_1^{\infty} f(x)\,dx - a_1 \leq \sum_{k=1}^{\infty} a_k \leq \int_1^{\infty} f(x)\,dx.
\]

4. Let \( \rho = \lim sup |a_n|^{1/n} \). Then \( \sum_{k=0}^{\infty} a_k \) converges absolutely if \( \rho < 1 \) and diverges if \( \rho > 1 \).
5. Let \( \rho = \lim \frac{|a_{n+1}|}{|a_n|} \) if it exists. Then \( \sum_{k=0}^{\infty} a_k \) converges absolutely if \( \rho < 1 \) and diverges if \( \rho > 1 \).
6. Let \( \{a_n\} \) be a sequence with \( 0 \leq a_{n+1} \leq a_n \). Then \( \sum_{k=0}^{\infty} (-1)^k a_k \) converges \( \iff a_n \to 0 \).

Let \( \sum_{k=0}^{\infty} c_k(x - a)^k \) be a power series and let

\[
R = \frac{1}{\lim sup |c_k|^{1/k}}.
\]

Then the power series converges on any interval \((r - a, r + a)\) where \( r < R \).

**Taylor’s formula:** If

\[
R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k
\]

then

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}
\]

for some \( c \) between \( a \) and \( x \).