#21.7 Assume that $f_n \to f$ uniformly and fix $\epsilon > 0$ then there exists an $N > 0$ such that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$. But $\bar{\rho}(f_n, f) = \sup_{x \in X} \{\min\{|f_n(x) - f(x)|, 1\}\} < \epsilon$ so $f_n \in B_{\rho}(f, \epsilon)$ so $f_n \to f$ in $\mathbb{R}^X$.

Now assume that $f_n \to f$ in $\mathbb{R}^X$. Then for for all $1 > \epsilon > 0$ there exists an $N > 0$ such that if $n > N$ then $f_n \in B_{\rho}(f, \epsilon)$. In particular $\min\{|f_n(x) - f(x)|, 1\} < \epsilon$ for all $x \in X$. So if $n > N$ we have $|f_n(x) - f(x)| < \epsilon$ and $f_n \to f$ uniformly.

#21.8 Since $f_n \to f$ uniformly there exists an $N_1 > 0$ such that if $n > N_1$ then $d(f_n(x_n), f(x_n)) < \epsilon/2$. Since the $f_n$ are continuous and the convergence is uniform by Theorem 21.6, $f$ is continuous and $f(x_n) \to f(x)$ (since $x_n \to x$). Therefore there exists an $N_2 > 0$ such that if $n > N_2$, $d(f(x_n), f(x)) < \epsilon/2$. Applying the triangle inequality we have $d(f_n(x_n) - f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \epsilon/2 + \epsilon/2 = \epsilon$ so $f_n(x_n) \to f(x)$.

#22.3 We first show the that for any continuous map $p : X \to Y$ if there is a continuous map $f : Y \to X$ such that $p \circ f$ is the identity map then $p$ is quotient map. Let $U \subset Y$ be a subset with $p^{-1}(U)$ open. Then $f^{-1}(p^{-1}(U))$ is open since $f$ is continuous but $f^{-1}(p^{-1}(U)) = (p \circ f)^{-1}(U) = U$ so this show that $U$ must be open in $Y$. By assumption $p$ is continuous so this shows that $p$ is a quotient map.

We’ll show that the quotient space is $\mathbb{R}$. Projection maps on product spaces are continuous so the restriction $q$ of $\pi_1$ to $A$ is also continuous. Define $f : \mathbb{R} \to A$ by $f(x) = (x, 0)$. Then $q \circ f$ is the identity so by the above paragraph $q$ is a quotient map so the quotient space is $\mathbb{R}$.

To show that $q$ is not a open take the open set $((-1, 1) \times (0, \infty)) \cap A = [0, 1) \times (0, \infty)$. The $q$-image of this open set is $[0, \infty)$ and is not open so $q$ is not an open map.

The set $\{(x, y) \in \mathbb{R}^2| y = 1/x\}$ is a closed subset of $A$ but its $q$-image is $(0, \infty)$ is not closed so $q$ is also not a closed map.
is a quotient map and the quotient space is $\mathbb{R}$. The quotient space will be $\mathbb{R}$ if and only if $g$ is continuous and $\circ f$ is the identity so $g$ is a quotient map and the quotient space is $\mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (x, 0)$. Both $f$ and $g$ are continuous and $g \circ f$ is the identity so $g$ is a quotient map and the quotient space is $\mathbb{R}$.

Let $Z = X \times Y - A \times B$ and let $C = \{(x, y) \in Z | z \notin A\}$ and $D = \{(x, y) \in Z | y \notin B\}$. Note that $Z = C \cup D$ since if $(x, y) \in Z$ then we must have either $x \notin A$ or $y \notin B$ (or possibly both). Let $(x_0, y_0) \in C$. Then any $(x_1, y_1) \in D$ is any the same connected component of $Z$ since the sets $\{x_0\} \times Y$ and $X \times \{y_1\}$ are connected subsets of $Z$ that have the point $(x_0, y_1)$ in common so there union is connected. Similarly every point in $C$ is in the same connected component as any point in $D$. This implies that $Z = C \cup D$ is connected.

Assume that $X$ is not connected and $A, B \subset X$ are a separation. Note that $p^{-1}(p(A)) = A$ since if $y \in P(A)$ then $p^{-1}(\{y\})$ must be entirely contained in $A$ since otherwise $p^{-1}(\{y\}) \cap A$ and $p^{-1}(\{y\}) \cap B$ would be a non-trivial separation of the connected set $p^{-1}(\{y\})$. Similarly $p^{-1}(p(B)) = B$. Since the sets $A$ and $B$ are open and $p$ is a quotient map this implies that $p(A)$ and $p(B)$ are open. They are also disjoint since $p^{-1}(p(A)) = A$ and $p^{-1}(p(B)) = B$ are disjoint. Therefore $p(A)$ and $p(B)$ are a non-trivial separation of $Y$, contradiction.

Assume that $Y \cup A$ has a non-trivial separation $C, D$. Note that $C$ and $D$ are open in the subspace topology for $Y \cup A$. Since $Y$ is connected it must be contained in $C$ or $D$. Let's say it is $C$. Since $D$ is disjoint from $C$, this implies that $D$ is contained in $A$. In particular, since $A$ is open in $X - Y$ is open in the subspace topology on $Y \cup A$ (and hence the subspace topology on $A$).

Define $g : [0, 1] \to \mathbb{R}$ by $g(x) = f(x) - x$. Then $g(0) = f(0) - 0 \geq 0$ and $g(1) = f(1) - 1 \leq 0$ so by the Intermediate Value Theorem there exists a $x \in [0, 1]$ such that $g(x) = 0$. But then $g(x) = f(x) - x = 0$ and $f(x) = x$ so $x$ is the desired fix point.

For a counterexample let $f(x) = x/2 + 1/2$. Then $f(x) = x$ if and only if $x = 1$ so $f$ doesn't have a fixed point on either $[0, 1)$ or $(0, 1)$.
#24.8(a) Yes. Let \((x_\alpha)\) and \((y_\alpha)\) be points in \(\prod X_\alpha\). Then for each \(\alpha\) there are paths \(\gamma_\alpha : [0,1] \to X_\alpha\) with \(\gamma_\alpha(0) = x_\alpha\) and \(\gamma_\alpha(1) = y_\alpha\). Define a path \(\gamma : [0,1] \to \prod X_\alpha\) by \(\gamma(t) = (\gamma_\alpha(t))\). Since each coordinate function is continuous, \(\gamma\) is continuous with \(\gamma(0) = (x_\alpha)\) and \(\gamma(1) = (y_\alpha)\). Therefore \(\gamma\) is a path from \((x_\alpha)\) to \((y_\alpha)\).

#24.8(b) No. Take the topologist's sine curve \(A = \{(x,y) \in \mathbb{R}^2 | y = \sin(1/x)\) and \(x > 0\}\) \(\subset \mathbb{R}^2\) is path connected but its closure is not.

#24.8(c) Yes. Let \(y_0\) and \(y_1\) be points in \(f(X)\). Then there exists \(x_i \in X\) with \(f(x_i) = y_i\). The composition of a path from \(x_0\) to \(x_1\) with \(f\) is a path from \(y_0\) to \(y_1\).

#24.8(d) Yes. Let \(x \in \cap A_\alpha\). Then for any \(x_0, x_1 \in \cup A_\alpha\) there are paths \(\gamma_0\) from \(x_0\) to \(x\) and \(\gamma_1\) from \(x\) to \(x_1\). The concatenation of these paths is a path from \(x_0\) to \(x_1\) so the union is path connected.

#24.10 Fix \(x_0 \in U\) and let \(A\) be the set of points \(x \in U\) such that there is a path in \(U\) from \(x_0\) to \(x\). We will show that \(A = U\) by showing that \(A\) is non-empty, open and closed. Clearly \(x_0 \in A\) so \(A\) is non-empty. For all \(x \in A\) there is a ball \(B_d(x, \epsilon)\) that is contained in \(U\). Balls are path connected so every \(y \in B_d(x, \epsilon)\) is in the same path connected component as \(y\) and hence as \(x_0\). Therefore \(B_d(x, \epsilon) \subset A\) and \(A\) is open. If \(x \in \bar{A}\) then every open neighborhood of \(x\) intersects \(A\). As before we have a ball \(B_d(x, \epsilon) \subset U\). Since this ball intersects \(A\) there is a path in \(U\) from \(x\) to a point in \(A\) and hence \(x \in A\) and \(A = \bar{A}\) is closed. Therefore \(A\) is non-empty, open and closed. Since \(U\) is connected, \(A = U\).