Each problem is worth 10 points. Show all of your work where appropriate for full credit.

1) Solve the following first order differential equations

a) $\frac{dy}{dx} = e^x; y(0) = 1$ Integrating up yields $y(x) = e^x + C$. The initial condition implies $C = 0$.

b) $\frac{dy}{dx} = y^2 x - y x; y(0) = 1$ Separating variables yields

$$\frac{1}{y(y - 1)} dy = x dx$$

$$\int \frac{-1}{y} + \frac{1}{y - 1} dy = \int x dx$$

$$-\ln(y) + \ln(y - 1) = x^2 + C$$

$$-\ln(y) + \ln(y - 1) = x^2 + C$$

$$1 - \frac{1}{y} = C e^{x^2}$$

$C = 0$

$$y(x) = 1$$

c) $\frac{dy}{dx} + \frac{1}{x}y = \sin(x); y(\pi) = 0$ Using an integrating factor $e^{\int \frac{1}{x} dx} = x$ we get the differential equation

$$(xy)' = x \sin(x)$$

Integrating up yields

$$xy = \int x \sin(x) \, dx + C = -x \cos(x) + \sin(x) + C$$

having integrated by parts. $C = -\pi$ so that the particular solution is

$$y(x) = -\cos(x) + \frac{1}{x} \sin(x) - \frac{\pi}{x}$$
2) The following autonomous ODE describes an excitable medium (i.e. cardiac tissue or a forest) without recovery.

\[
\frac{dV}{dt} = V(1 - V)(V - a) \quad 0 < a < \frac{1}{2}
\]

a) In general, what is a critical point?
A critical point of an autonomous ODE \( \frac{dy}{dx} = f(y) \) are the \( y \)-values such that \( f(y) = 0 \). At these \( y \)-values then \( \frac{dy}{dx} = 0 \) so that the solution doesn’t change (i.e. the solution is a fixed point).

b) What are the critical points of the given autonomous ODE?
\( V(1 - V)(V - a) = 0 \) for \( V = 0, a, 1 \).

c) Plot the phase line and determine the stability of the critical points.

\[
V = 0 \text{ - stable; } V = a \text{ - unstable; } V = 1 \text{ - stable}
\]

d) Why is \( V = a \) considered a threshold?
\( V = a \) is a threshold, because if we start just below \( V = a \), then we go to \( V = 0 \). However, if we start just above \( V = a \), then we go to \( V = 1 \).

e) (Extra Credit) Find the implicit general solution of the given ODE. Separating variables and integrating yields

\[
\int \frac{1}{V(1 - V)(V - a)} \, dV = \int \, dt + C
\]

\[
\int \frac{-1/a}{V} + \frac{1/(1 - a)}{1 - V} + \frac{1/(a(1 - a))}{V - a} \, dV = t + C
\]

\[
-1/a \ln(V) - 1/(1 - a) \ln(1 - V) + 1/(a(1 - a)) \ln(V - a) = t + C
\]

\[
\frac{V^{1/(a(1 - a))}}{V^{1/(1 - a)}} = Ce^t
\]
\( \frac{dy}{dx} = xy ; y(0) = -1 \)

a) Prove the existence and uniqueness of the given IVP.

Since \( f(x, y) = xy \) is continuous for all values of \( x \) and \( y \) it is continuous in a neighborhood of the initial value, and so we are guaranteed existence by the existence/uniqueness theorem. Since \( \frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} xy = x \), is also continuous for all values of \( x \) and \( y \) including the initial value, we are also guaranteed uniqueness by the existence/uniqueness theorem.

b) Plot the direction field for the given ODE, and sketch the unique solution of the IVP.

![Direction Field](image.png)

Solid curve is the solution of the IVP. The dashed curves are isoclines.

c) Use Euler’s Method to approximate \( y \) at \( x = 1 \) using a stepsize of \( h = 0.5 \).

\[
\begin{array}{c|c}
 x & y (y_{n+1} = y_n + h(x_ny_n)) \\
x_0 = 0 & y_0 = -1 \\
x_1 = 0.5 & y_1 = -1 + 0.5(0(-1)) = -1 \\
x_2 = 1 & y_2 = -1 + 0.5(0.5(-1)) = -1.25 \\
\end{array}
\]
4) a) If it exists, find the inverse of the given matrix.

i) \( A = \begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \)

\[ |A| = 1 \cdot 8 - (-2) \cdot (-4) = 0 \]

so \( A \) is not invertible.

ii) \( B = \begin{bmatrix} 2 & 7 & 4 \\ 1 & 3 & 2 \\ 2 & 6 & 5 \end{bmatrix} \)

We do elementary row operations to check the determinant, but since we would need to do these operations to find the inverse anyway, we augment the matrix at the start.

\[
[B|I] = \begin{bmatrix} 2 & 7 & 4 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 7 & 4 & 1 & 0 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{bmatrix} \rightarrow_2 \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{bmatrix} \rightarrow_2 \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{bmatrix} \rightarrow_2 \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{bmatrix} \rightarrow_3 \begin{bmatrix} 1 & 0 & 0 & -3 & 11 & -2 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{bmatrix} \rightarrow_3 \begin{bmatrix} 1 & 0 & 0 & -3 & 11 & -2 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{bmatrix} \rightarrow_3 \begin{bmatrix} -3 & 11 & -2 \\ 1 & -2 & 0 \\ 0 & -2 & 1 \end{bmatrix}
\]

Here we notice that we have used elementary row ops. to reduce \( B \) to a triangular matrix with nonzero diagonal elements so that the determinant of \( B \) is nonzero and the inverse (which we’re in the process of finding) exists. Note that the fact the determinant is nonzero is not affected by the row swap. Why not?

So

\[
B^{-1} = \begin{bmatrix} -3 & 11 & -2 \\ 1 & -2 & 0 \\ 0 & -2 & 1 \end{bmatrix}
\]

iii) \( C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ -2 & 3 & -4 \end{bmatrix} \)

Since \(-R_2 = R_3\), performing the row op. \( R_2 + R_3 \rightarrow R_3 \) yields a zero row, and thus, a zero determinant implying no inverse exists.
b) Find the solution set of the given linear system.

i) \[ \begin{align*}
    x_1 - 2x_2 &= 0 \\
    -4x_1 + 8x_2 &= 0
\end{align*} \]

From 4a)i) we know that the coefficient matrix is not invertible so we either have no solutions or infinitely many solutions, but this is also a homogeneous system, so that we must have an infinite number of solutions. Reducing the system and letting \( x_2 = t \) a free variable, we get the solution set is \( (x_1 = 2t, x_2 = t) \) for \( t \in R \).

ii) \[ \begin{align*}
    2x + 7y + 4z &= 0 \\
    x + 3y + 2z &= 1 \\
    2x + 6y + 5z &= 4
\end{align*} \]

From 4a)ii) we know that the inverse of the coefficient matrix exists so our solution is

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = B^{-1} \begin{pmatrix}
    0 \\
    1 \\
    4
\end{pmatrix}
\]

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = \begin{pmatrix}
    -3 & 11 & -2 \\
    1 & -2 & 0 \\
    0 & -2 & 1
\end{pmatrix} \begin{pmatrix}
    0 \\
    1 \\
    4
\end{pmatrix}
\]

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = \begin{pmatrix}
    3 \\
    -2 \\
    2
\end{pmatrix}
\]

iii) \[ \begin{align*}
    x_1 + x_2 + x_3 &= 1 \\
    -2x_1 + 3x_2 - 4x_3 &= 2 \\
    2x_1 - 3x_2 + 4x_3 &= 3
\end{align*} \]

Again, since no inverse exists (from 4a)iii) ), we know we either have no solutions or infinitely many solutions. Performing the row op. \( R_2 + R_3 \rightarrow R_3 \) yields the inconsistent system.

\[
\begin{bmatrix}
    1 & 1 & 1 & | & 1 \\
    -2 & 3 & -4 & | & 2 \\
    0 & 0 & 0 & | & 5
\end{bmatrix}
\]

Thus, there is no solution.