8) If \( y_1 = \cos(x) - \cos(2x) \) and \( y_2 = \sin(x) - \cos(2x) \), then \( y_1' = -\sin(x) + 2\sin(2x), \ y_1'' = -\cos(x) + 4\cos(2x) \) and \( y_2' = \cos(x) + 2\sin(2x), \ y_2'' = -\sin(x) + 4\cos(2x) \). Plugging into the left hand side of the ODE \( y_1'' + y_1 = 3\cos(2x) \) we get
\[
y_1'' + y_1 = -\cos(x) + 4\cos(2x) + (\cos(x) - \cos(2x)) = 3\cos(2x)
\]
and
\[
y_2'' + y_2 = -\sin(x) + 4\cos(2x) + (\sin(x) - \cos(2x)) = 3\cos(2x).
\]
So \( y_1 \) and \( y_2 \) solve the ODE \( y_1'' + y_1 = 3\cos(2x) \).

16) Substituting \( y = e^{rx} \) into the ODE \( 3y'' + 3y' - 4y = 0 \) yields the equation
\[
3r^2 e^{rx} + 3re^{rx} - 4e^{rx} = 0.
\]
Dividing this equation by (the nonzero) \( e^{rx} \) yields a quadratic equation for \( r \)
\[
3r^2 + 3r - 4 = 0
\]
which we solve by the quadratic equation to get \( r = \frac{3 \pm \sqrt{57}}{6} \).

18) We check that \( y(x) = Ce^{2x} \) satisfies the ODE \( y' = 2y \). Taking the derivative of \( y \) we get \( y' = 2Ce^{2x} \), so we see that \( y' \) is exactly \( 2y \). To satisfy the initial condition \( y(0) = 3, \ y(0) = Ce^{2 \cdot 0} = C \cdot 1 = C = 3 \), and \( C \) must be 3.

24) We check that \( y(x) = x^3(C + \ln(x)) \) satisfies the ODE \( xy' - 3y = x^3 \). Taking the derivative of \( y \) we get \( y' = 3x^2(C + \ln(x)) + x^2 \left( \frac{1}{x} \right) = 3x^2(C + \ln(x)) + x^2 \), so we see that
\[
xy' - 3y = x(3x^2(C + \ln(x)) + x^2) - 3(x^3(C + \ln(x)))
\]
\[
= 3x^3C + 3x^3\ln(x) + x^3 - 3x^3C - 3x^3\ln(x)
\]
\[
= x^3,
\]
which is exactly the right hand side. To satisfy the initial condition \( y(1) = 17, \ y(0) = 1^3(C + 0) = 17, \) and \( C \) must be 17.

28) The slope of the line through the two points \((x,y)\) and \((x/2,0)\) is given by
\[
y' = \frac{y-0}{x-x/2} = \frac{2y}{x}
\]
so that the function which has a tangent line at \((x,y)\) and goes through \((x/2,0)\) also solves the ODE \( 2y' = x \).

32) The time rate of change of a population \( P \) being proportional to the square root of the population translates to the equation
\[
\frac{dP}{dt} = k\sqrt{P}
\]
where \( k \) is the constant of proportionality and has units of \(1/\text{time}\).

6) To find the general solution to \( y' = x\sqrt{x^2+9} \) we integrate up both sides so that \( y = \int (x\sqrt{x^2+9} \ dx) + C \). If we make the substitution \( u = x^2 + 9 \) so that \( du = 2x \ dx \), then \( \int (x\sqrt{x^2+9} \ dx) = \frac{1}{2} \int \sqrt{u} \ du = \frac{1}{3}u^{3/2} \). From this we get that \( y = \frac{1}{3}(x^2 + 9)^{3/2} + C \). We use \( y(-4) = 0 \) to get that \( C = -\frac{125}{3} \) and the particular solution is \( y = \frac{1}{3}(x^2 + 9)^{3/2} - \frac{125}{3} \).

8) To find the general solution to \( y' = \cos(2x) \) we integrate up both sides so that \( y = \int \cos(2x) \ dx + C = \frac{1}{2}\sin(2x) + C \). We use \( y(0) = 1 \) to get that \( C = 1 \) and the particular solution is \( y = \frac{1}{2}\sin(2x) + 1 \).

14) To find the position function \( x(t) \) from acceleration we integrate up twice, first to get velocity and then to get position.
\[
a(t) = v'(t)
\]
so that
\[
v(t) = \int a(t) \ dt + C = \int 2t + 1 \ dt + C = t^2 + t + C.
\]
Since \( v(0) = -7, \ C = -7 \). Now
\[
v(t) = x'(t)
\]
so that
\[
x(t) = \int v(t) \ dt + C = \int t^2 + t - 7 \ dt + C = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 7t + C.
\]
Since \( x(0) = 4, \ C = 4 \), so that the particular solution is
\[
x(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 7t + 4.
\]
20) To find how long it takes a ball dropped (with 0 initial velocity) from a 400ft high building to hit the ground we consider finding position from a constant gravity acceleration. First we integrate to find the velocity function $v(t) = -32t$. Integrating the velocity equation to get position yields $x(t) = -16t^2 + 400$. The ball hits the ground when $x = -16t^2 + 400 = 0$ or when $t = 5$sec. At this time the velocity $v(5) = -32(5) = -160$ft/sec.

36) Considering the preamble to Example 4 and the velocity of the river is $v_R = v_0 \left(1 - \frac{x^4}{a^4}\right)$ we get the ODE

$$y' = \frac{v_0}{v_S} \left(1 - \frac{x^4}{a^4}\right).$$

We can integrate this up to get

$$y = \frac{v_0}{v_S} \left(x - \frac{1}{5} \frac{x^5}{a^4}\right) + C$$

Starting from the left bank gives us the condition $y(-a) = 0$, so that $C = \frac{v_0}{v_S} \frac{4}{5} a$. When the swimmer reaches the other side ($x = a$), he is $y(a) = \frac{v_0}{v_S} \frac{8}{5} a$ down stream. Using $v_0 = 9$mi/h, $v_S = 3$mi/h, and $a = \frac{1}{2}$, we get that $y\left(\frac{1}{2}\right) = 2.4$mi.
22) Since \( f(x, y) = x \ln(y) \) is continuous on the rectangle given by \( \{-\infty < x < \infty, 0 < y \} \) and the initial condition \( y(1) = 1 \) falls into this rectangle, the ODE \( y' = f(x, y) \) is guaranteed a solution exists. Furthermore since \( \frac{\partial f}{\partial y} = \frac{x}{y} \) is continuous everywhere except when \( y = 0 \), we're also guaranteed a unique solution for a while.

28) Since \( f(x, y) = \frac{y-1}{y} \) is continuous on the two rectangles given by \( \{-\infty < x < \infty, 0 < y \} \) and \( \{-\infty < x < \infty, y < 0 \} \), but the initial condition \( y(1) = 0 \) falls between these rectangles, the ODE \( y' = f(x, y) \) is not guaranteed a solution exists.

32) \( y(x) = 0 \) is a solution for all \( x \)-values. We can solve this ODE by separation of variables to get the solution \( y(x) = x^3 \) (as discussed in class). Since the \( \frac{\partial}{\partial y} 3y^{2/3} = 2y^{-1/3} \) is not continuous at \( y = 0 \), the theorem does not guarantee us uniqueness. Thus, the two solutions do not conflict with the theorem.

CPA  a) Plugging in \( y = ax + b \) into the ODE \( y' = \sin(x - y) \) yields the equation

\[
 a = \sin((1 - a)x + b).
\]

Since the left hand side is constant, the right hand side must also be constant. This is only true when \( (1 - a) = 0 \) or \( a = 1 \). This leaves us with the equation

\[
 1 = \sin(b)
\]

to be solved for \( b \). This yields \( b = n\pi/2 \) for \( n = ..., -2, -1, 0, 1, 2, ... \)

b) No, we would need \( C \rightarrow \infty \) for the initial condition \( y(\pi/2) = 0 \) to satisfy \( y(x) = x - 2 \tan^{-1} \left( \frac{x - C}{\sqrt{x^2 - C^2}} \right) \).

CPB Similar arguments to CPA.
2) Separating the variables of the ODE \( y' + 2xy^2 = 0 \) yields the equation

\[
\frac{1}{y^2} \, dy = -2x \, dx
\]

which we integrate up to get

\[
-\frac{1}{y} = -x^2 + C.
\]

Solving for \( y \) explicitly we get \( y = \frac{1}{x^2 + C} \) (\( C \) has gone to \(-C \) because it is a general constant).

12) Separating the variables of the ODE \( yy' = x(y^2 + 1) \) yields the equation

\[
\frac{y}{y^2 + 1} \, dy = x \, dx
\]

which we integrate up (using \( u \)-substitution) to get

\[
\frac{1}{2} \ln(y^2 + 1) = \frac{1}{2} x^2 + C.
\]

Solving for \( y^2 \) we get the implicit general solution \( y^2 = Ce^{x^2} - 1 \).

20) Separating the variables of the ODE \( y' = 3x^2(y^2 + 1) \) yields the equation

\[
\frac{1}{y^2 + 1} \, dy = 3x^2 \, dx
\]

which we integrate up (using \( \int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} \)) to get

\[
\tan^{-1}(y) = x^3 + C.
\]

Solving for \( y \) we get the explicit general solution \( y = \tan(x^3 + C) \). Using the initial condition \( y(0) = 1 \), we get that \( C = \tan^{-1}(1) = \pi/4 \). The particular solution is then \( y(x) = \tan(x^3 + \pi/4) \).

26) Separating the variables of the ODE \( y' = 2xy^2 + 3x^2y^2 \) yields the equation

\[
\frac{1}{y^2} \, dy = 2x + 3x^2 \, dx
\]

which we integrate up to get

\[
-\frac{1}{y} = x^2 + x^3 + C.
\]

Solving for \( y \) we get the explicit general solution \( y = \frac{1}{x^2 + x^3 + C} \). Using the initial condition \( y(1) = -1 \), we get that \( 1 = \frac{1}{2 + C} \), so that \( C = -1 \). The particular solution is then \( y(x) = \frac{1}{x^2 + x^3 - 1} \).

42) Given the initial value problem for barometric pressure in terms of altitude, \( x, p' = -0.2p \) with initial condition \( p(0) = 29.92 \), we are asked to find the pressure at 10,000ft and at 30,000ft. We separate variables to solve for \( p \) and get that \( p(x) = 29.92e^{-0.2x} \). Plugging in our two altitudes gives us \( p(10,000 \text{ft}) = 1.9mi = 29.92e^{-0.2\cdot1.9} \approx 20.5 \) and \( p(30,000 \text{ft}) = 5.7mi = 29.92e^{-0.2\cdot5.7} \approx 9.6 \). If people cannot survive at \( p < 15 \) then solving for \( x \) at this \( p \)-value tells us the altitude at which this occurs (should be between 10,000ft and 30,000ft from our previous calculation). Solving for \( x \) in \( 15 = 29.92e^{-0.2x} \) yields \( x \approx 3.45mi \approx 18,200 \text{ft} \).

CP Consider the ODE

\[
\frac{dx}{dt} = \frac{m}{n} x - \frac{1}{n} x^2.
\]

Letting \( m = 9 \) and \( n = 7 \), we get the ODE

\[
\frac{dx}{dt} = \frac{1}{7} (9x - x^2).
\]

As \( t \to \infty \) it appears that the population approaches 9.

b) Maple gives the general solution as \( x(t) = \frac{1}{1 + 9e^{-\frac{t}{7}}} \). As \( t \to \infty \), \( x \to 9 \) as expected.

c) We ask the question how long does it take for a population to grow by 80%? Letting \( x(0) = x_0 \) we get the particular population \( x(t) = \frac{9x_0}{x_0 + (9 - x_0)e^{-\frac{t}{7}}} \). The population is \( x = 0.8x_0 \) when \( t = -7/\ln(\frac{9}{8 - x_0}) \). We can now plug in whatever initial population we like to get a specific \( t \).