output is 4, it is impossible to tell which was the input. A graph shows that this function fails the horizontal line test at almost all values in its range.

In addition, Algorithm 1.1 might fail because the algebra is impossible. Step 2 requires solving an equation. Many equations cannot be solved algebraically.

Example 1.2.27
A Function with an Inverse That Is Impossible to Compute Algebraically

Consider the function

\[ f(x) = x^5 + x + 1 \]

The graph satisfies the horizontal line test (see Figure 1.2.32). We try to find the inverse \( f^{-1}(y) \) as follows:

1. Set \( y = x^5 + x + 1 \).

2. Try to solve for \( x \). Even with the cleverest algebraic tricks, this is impossible (a remarkable theorem, proved by the French mathematician Evariste Galois when he was just 20 years old, assures us that there is no formula for the solution of a general polynomial with degree greater than 4).

3. Give up.

In mathematical modeling, however, it is often more important to know that something exists (such as the inverse in this case) than to be able to write down a formula. We will later learn a method to compute this inverse numerically, with a computer (Section 3.8).

Summary
Quantitative science is built upon measurements, and mathematics provides the methods for describing and thinking about measurements and relations between them. Variables describe measurements that change during the course of an experiment, and parameters describe measurements that remain constant during an experiment but might change between different experiments. Functions describe relations between different measurements when a single output is associated with each input; they can be recognized graphically with the vertical line test. New functions are built by combining functions through addition, multiplication, and composition. In functional composition, the output of the inner function is used as the input of the outer function. Many functions do not commute, meaning that composing the functions in a different order gives a different result. Finally, we can use the horizontal line test to check whether a functions has an inverse. If it does, the inverse can be used to compute the input from the output.

1.2 Exercises

Mathematical Techniques
1-2. Give mathematical names to the measurements in the following situations, and identify the variables and parameters.

1. A scientist measures the density of wombats at three altitudes: 500 m, 750 m, and 1000 m. He repeats the experiment in 3 different years, with rainfall of 30 cm in the first year, 50 cm in the second, and 60 cm in the third.

2. A scientist measures the density of bandicoots at three altitudes: 500 m, 750 m, and 1000 m. She repeats the experiment in three different years that have different densities of wombats, which compete with bandicoots. The density is 10 wombats per square kilometer in the first year, 20 wombats per square kilometer in the second, and 15 wombats per square kilometer in the third.

3-6. Compute the values of the following functions at the points indicated and sketch a graph.

3. \( f(x) = x + 5 \) at \( x = 0, x = 1, \) and \( x = 4 \)

4. \( g(y) = 5y \) at \( y = 0, y = 1, \) and \( y = 4 \)

5. \( h(z) = \frac{1}{z} \) at \( z = 1, z = 2, \) and \( z = 4 \)

6. \( F(r) = r^2 + 5 \) at \( r = 0, r = 1, \) and \( r = 4 \)
7-10 • Graph the given points and say which point does not seem to fall on the graph of a simple function that describes the other four.
7.  (0, -1), (1, 1), (2, 2), (3, 5), (4, 7)
8.  (0, 8), (1, 10), (2, 8), (3, 6), (4, 4)
9.  (0, 2), (1, 3), (2, 6), (3, 11), (4, 12)
10. (0, 30), (1, 25), (2, 15), (3, 12), (4, 10)

11-14 • Evaluate the following functions at the given algebraic arguments.
11.  f(x) = x + 5 at x = a, x = a + 1, and x = 4a
12.  g(y) = 5y at y = x², y = 2x + 1, and y = 2 - x
13.  h(z) = \frac{1}{\sqrt{z}} at z = \frac{c}{3}, z = \frac{c}{5}, and z = c + 1
14.  F(r) = r² + 5 at r = x + 1, r = 3x, and r = \frac{1}{x}

15-16 • Sketch graphs of the following relations. Is there a more convenient order for the arguments?
15.  A function whose argument is the name of a state and whose value is the highest altitude in that state.

<table>
<thead>
<tr>
<th>State</th>
<th>Highest Altitude (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>California</td>
<td>14,491</td>
</tr>
<tr>
<td>Idaho</td>
<td>12,662</td>
</tr>
<tr>
<td>Nevada</td>
<td>13,143</td>
</tr>
<tr>
<td>Oregon</td>
<td>11,239</td>
</tr>
<tr>
<td>Utah</td>
<td>13,528</td>
</tr>
<tr>
<td>Washington</td>
<td>14,410</td>
</tr>
</tbody>
</table>

16.  A function whose argument is the name of a bird and whose value is the average length of that bird.

<table>
<thead>
<tr>
<th>Bird</th>
<th>Length (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooper’s hawk</td>
<td>50</td>
</tr>
<tr>
<td>Goshawk</td>
<td>66</td>
</tr>
<tr>
<td>Sharp-shinned hawk</td>
<td>35</td>
</tr>
</tbody>
</table>

21-24 • For each of the following pairs of functions, graph each component piece. Compute the value of the product at x = -2, x = -1, x = 0, x = 1, and x = 2 and graph the result.
21.  f(x) = 2x + 3 and g(x) = 3x - 5
22.  f(x) = 2x + 3 and h(x) = -3x - 12
23.  g(x) = x² + 1 and G(x) = x + 1
24.  F(x) = x² + 1 and H(x) = -x + 1

25-28 • Find the inverse of each of the following functions when an inverse exists. In each case, compute the output at an input of 1.0, and show that the inverse undoes the action of the function.
25.  f(x) = 2x + 3
26.  g(x) = 3x - 5
27.  F(x) = y² + 1
28.  F(y) = y² + 1 for y ≥ 0

29-32 • Graph each of the following functions and its inverse if it exists. Mark the given point on the graph of each function.
29.  f(x) = 2x + 3. Mark the point (1, f(1)) on the graphs of f and the corresponding point on f⁻¹ (based on Exercise 25).
30.  g(x) = 3x - 5. Mark the point (1, g(1)) on the graphs of g and the corresponding point on g⁻¹ (based on Exercise 26).
31.  F(y) = y² + 1. Mark the point (1, F(1)) on the graphs of F and the corresponding point on F⁻¹ (based on Exercise 27).
32.  F(y) = y² + 1 for y ≥ 0. Mark the point (1, F(1)) on the graphs of F and the corresponding point on F⁻¹ (based on Exercise 28).

33-36 • Find the compositions of the given functions. Which pairs of functions commute?
33.  f(x) = 2x + 3 and g(x) = 3x - 5
34.  f(x) = 2x + 3 and h(x) = -3x - 12
35.  F(x) = x² + 1 and G(x) = x + 1
36.  F(x) = x² + 1 and H(x) = -x + 1

Applications
37-40 • Describe what is happening in the graphs shown.
37.  A plot of cell volume against time in days.
36. A plot of a Pacific salmon population against time in years.

39. A plot of the average height of a population of trees plotted against age in years.

46. The number of cancerous cells \( c \) as a function of radiation dose \( r \) (measured in rads) is

\[
c = r - 4
\]

for \( r \) greater than or equal to 5, and is zero for \( r \) less than 5. Suppose \( r \) ranges from 0 to 10. What is happening at \( r = 5 \) rads?

47. Insect development time \( A \) (in days) obeys \( A = 40 - \frac{T}{2} \)

where \( T \) represents temperature in °C for \( 10 \leq T \leq 40 \). Which temperature leads to the most rapid development?

48. Tree height \( h \) (in meters) follows the formula

\[
h = \frac{100a}{100 + a}
\]

where \( a \) represents the age of the tree in years. The formula is valid for any positive value of \( a \), which ranges from 0 to 1000. How tall would this tree get if it lived forever?

49-52 * Consider the following data describing the growth of an tadpole.

<table>
<thead>
<tr>
<th>Age, ( a ) (days)</th>
<th>Length, ( L ) (cm)</th>
<th>Tail Length, ( T ) (cm)</th>
<th>Mass, ( M ) (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.5</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td>1.0</td>
<td>3.0</td>
<td>0.9</td>
<td>3.0</td>
</tr>
<tr>
<td>1.5</td>
<td>4.5</td>
<td>0.8</td>
<td>6.0</td>
</tr>
<tr>
<td>2.0</td>
<td>6.0</td>
<td>0.7</td>
<td>12.0</td>
</tr>
<tr>
<td>2.5</td>
<td>7.5</td>
<td>0.6</td>
<td>24.0</td>
</tr>
<tr>
<td>3.0</td>
<td>9.0</td>
<td>0.5</td>
<td>48.0</td>
</tr>
</tbody>
</table>

49. Graph length as a function of age.
50. Graph tail length as a function of age.
51. Graph tail length as a function of length.
52. Graph mass as a function of length, and then graph length as a function of mass. How do the two graphs compare?

53-56 * The following series of functional compositions describe connections between several measurements.

53. The number of mosquitos \( (M) \) that end up in a room is a function of how much the window is open \( (W, \) in square centimeters) according to \( M(W) = 5W + 2 \). The number of bites \( (B) \) depends on the number of mosquitos according to \( B(M) = 0.5 M \). Find the number of bites as a function of how much the window is open. How many bites would you get if the window were 10 cm² open?

54. The temperature of a room \( (T, \) in degrees Celsius) is a function of how much the window is open \( (W, \) in square centimeters) according to \( T(W) = 40 - 0.2W \). How long you sleep \( (S, \) measured in hours) is a function of the temperature according to \( S(T) = 14 - \frac{T}{5} \). Find how long you sleep as a function of how much the window is open. How long would you sleep if the window were 10 cm² open?