Exercises for Section 4.6: Mean Value Theorem

1. Prove that every convex set in $\mathbb{R}^n$ is connected.

   **Proof.** Let $A \subset \mathbb{R}^n$ be a convex set, and let $x, y \in A$. Then the path $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ given by $\gamma(t) = ty + (1-t)x$ is in $A$ because $A$ is convex, and connects $x$ to $y$. Therefore $A$ is path-connected. □

2. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function which is continuously differentiability on a convex set $C \subset \mathbb{R}^n$, and $df_x = 0$ for all $x \in C$, show that $f$ is constant on $C$.

   **Proof.** This follows from the Corollary to the Mean Value Theorem. Let $a, b \in C$. Then $df_x = 0$ for all $x \in C$ implies that $\|f(b) - f(a)\| = 0$. Therefore, $f(a) = f(b)$. So $f$ is constant on $C$. □

3. Give an example of a connected set in $\mathbb{R}^n$ which is not convex. Is it possible for a connected set in $\mathbb{R}$ to be non-convex?

   **Solution.** See the example in the notes.

4. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be **convex** if for every $a, b \in \mathbb{R}$ the secant line joining $(a, f(a))$ to $(b, f(b))$ lies above the graph of $f$. That is, for all $t \in [0, 1],

   $$f(tb + (1-t)a) \leq tf(b) + (1-t)f(a).$$

   (In calculus, we called such a function “concave up.”)

   Show that $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex (as a function) if and only if the set $A = \{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ is convex (as a set).

   **Proof.** First suppose $f$ is a convex function. Let $(x_1, y_1)$ and $(x_2, y_2)$ be two points in $A$. Notice that $y_1 \geq f(x_1)$ and $y_2 \geq f(x_2)$. Therefore, the line segment from $(x_1, y_1)$ to $(x_2, y_2)$ lies above the line segment from $(x_1, f(x_1))$ to $(x_2, f(x_2))$. Thus we conclude that the line segment is entirely within $A$, hence $A$ is a convex set.

   Now suppose $A$ is a convex set. Let $a, b \in \mathbb{R}$. Then $(a, f(a))$ and $(b, f(b))$ are points in $A$. Therefore, since $A$ is convex, the line segment between these two points must be in $A$. In other words, this line segment must lie above the graph of $f$. Hence, $f$ is a convex function. □

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}^m$ be continuously differentiable, and $a \in \mathbb{R}$. Show that $df_a(1) = f'(a)$. More generally, show that $df_a(s) = s \cdot f'(a)$.

   **Proof.** These follow directly from the definition of the derivative. □
6. Let $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that for each unit vector $u \in \mathbb{R}^n$ ($\|u\| = 1$), the directional derivative $\partial f_{a+tu}(u)$ exists for $t \in [0,1]$. Prove that

$$f(a+u) - f(a) = \partial f_{a+\tau u}(u)$$

for some $\tau \in (0,1)$. (Warning: $f$ might not be differentiable at $a$ even though all the directional derivatives exist.)

**Proof.** Let $u$ be a unit vector in $\mathbb{R}^n$. Define the path $\gamma : [0,1] \to \mathbb{R}^n$ via $\gamma(t) = a + tu$, and let $g : [0,1] \to \mathbb{R}$ by $g(t) = f(\gamma(t))$.

Now we claim that $g$ is differentiable. Compute

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h} = \lim_{h \to 0} \frac{f(a + (t+h)u) - f(a + tu)}{h} = \lim_{h \to 0} \frac{f((a+tu) + hu) - f(a + tu)}{h} = \partial f_{a+tu}(u).$$

Because the directional derivative of $f$ exists, then $g$ is differentiable. Moreover, by the (single-variable) Mean Value Theorem, there is some point $\tau \in [0,1]$ such that

$$g(1) - g(0) = g'(\tau)$$

$$f(a + u) - f(a) = \partial f_{a+\tau u}(u).$$