Exercises for Section 4.4: The Derivative

1. Find expressions for the partial derivatives of the following functions:
   
   (a) \( F(x, y) = f(g(x)k(y), g(x) + h(y)) \).
   
   (b) \( F(x, y, z) = f(g(x + y), h(y + z)) \).
   
   (c) \( F(x, y, z) = f(x^y, y^z, z^x) \).
   
   (d) \( F(x, y) = f(x, g(x), h(x, y)) \).

2. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \). For \( v \in \mathbb{R}^n \), the limit
   \[
   \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} ,
   \]
   if it exists, is denoted \( \partial f_a(v) \), and is called the **directional derivative** of \( f \) at \( a \) in the direction \( v \).

   (a) Show that \( \partial f_a(e_i) = \frac{\partial f}{\partial x^i}(a) \).

   **Proof.**
   \[
   \partial f_a(e_i) = \lim_{t \to 0} \frac{f(a + te_i) - f(a)}{t} = \lim_{t \to 0} \frac{f(a^1, \ldots, a^i + t, \ldots, a^n) - f(a^1, \ldots, a^n)}{t} = \frac{\partial f}{\partial x^i}(a).
   \]

   (b) Show that \( \partial f_a(cv) = c \cdot \partial f_a(v) \). (Scalar multipliers come out).

   **Proof.**
   \[
   \partial f_a(cv) = \lim_{t \to 0} \frac{f(a + t(cv)) - f(a)}{t} = \lim_{t \to 0} \frac{f(a + (ct)v) - f(a)}{t} = \lim_{\tau \to 0} \frac{f(a + \tau v) - f(a)}{\tau/c} \text{ (where } \tau = ct) = c \cdot \partial f_a(v).
   \]

   (c) If \( f \) is differentiable at \( a \), show that \( \partial f_a(v) = df_a(v) \), so therefore \( \partial f_a(u + v) = \partial f_a(u) + \partial f_a(v) \).

   **Proof.** Because \( f \) is differentiable at \( a \), we know that
   \[
   \lim_{h \to 0} \frac{|f(a + h) - f(a) - df_a(h)|}{\|h\|} = 0.
   \]
But this limit is 0 regardless of the way in which the vector \( h \) approaches 0. Let \( h = tv \). Then
\[
0 = \lim_{t \to 0} \frac{|f(a + tv) - f(a) - df_a(tv)|}{|tv|} = \lim_{t \to 0} \frac{|f(a + tv) - f(a) - t df_a(v)|}{|t||v|}.
\]
Multiplying both sides by \(|v|\) yields
\[
0 = \lim_{t \to 0} \frac{|f(a + tv) - f(a) - t df_a(v)|}{|t|}.
\]
Now we have a ratio of real numbers. The only way for the above limit to be 0 is if
\[
\lim_{t \to 0} f(a + tv) - f(a) - t df_a(v) = 0.
\]
We can separate this into
\[
df_a(v) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} = \partial f_a(v).
\]
Note that \( \partial f_a(v) \) may exist for every direction \( v \), but \( f \) still fail to be differentiable, as is seen in Problem 3, below.

3. Let \( f \) be defined as in Problem ???. Show that \( \partial f_{(0,0)}(x) \) exists for all directions \( x \), but if \( g \neq 0 \), then
\[
\partial f_{(0,0)}(u + v) = \partial f_{(0,0)}(u) + \partial f_{(0,0)}(v)
\]
is not true for some \( u, v \in \mathbb{R}^2 \). Therefore, by Problem 2, above, \( f \) must not be differentiable.

**Proof.** Recall the definition of \( f \). Let \( g \) be a continuous real-valued function on the unit circle,
\[
S^1 = \{ x \in \mathbb{R}^2 : \|x\| = 1 \}
\]
such that \( g(0,1) = g(1,0) = 0 \) and \( g(-x) = -g(x) \). Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
\|x\| \cdot g \left( \frac{x}{\|x\|} \right), & \text{if } x \neq 0, \\
0, & \text{if } x = 0.
\end{cases}
\]
Let \( x \) be any direction. Since scalars come out of the directional derivative, we may assume \( \|x\| = 1 \). Then,
\[
\partial f_{(0,0)}(x) = \lim_{t \to 0} \frac{f(tx) - f(0)}{t} = \lim_{t \to 0} \frac{|t|g \left( \frac{tx}{|t|} \right)}{t}.
\]
Note that \( g(-x) = -g(x) \), so we are left with
\[
\partial f_{(0,0)}(x) = g(x).
\]
If \( g \neq 0 \) then there is some point \((a, b) \in S^1\) such that \( g(a, b) \neq 0 \). However, let \( u = (a,0) \) and \( v = (0,b) \). Then \( \partial f_0(u + v) = g(a,b) \); while \( \partial f_0(u) = a \partial f_0(1,0) = ag(1,0) = 0 \) and \( \partial f_0(v) = b \partial f_0(0,1) = bg(0,1) = 0 \). Therefore
\[
\partial f_0(u + v) \neq \partial f_0(u) + \partial f_0(v),
\]
hence \( f \) is not differentiable at 0.
4. (a) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined as

\[
f(x) = \begin{cases} 
  x^2 \sin(1/x), & \text{if } x \neq 0 \\
  0, & \text{if } x = 0
\end{cases}
\]

Show that \( f \) is differentiable at 0, but \( f' \) is not continuous at 0.

**Proof.** The claim is that \( df_0 = 0 \). Check:

\[
\lim_{h \to 0} \frac{|f(0 + h) - f(0) - 0|}{|h|} = \lim_{h \to 0} \frac{|h^2 \sin(1/h)|}{|h|} = \lim_{h \to 0} |h \sin(1/h)|.
\]

But this limit is indeed 0 by the Squeeze Theorem (bounded by the functions \(-|h|\) and \(|h|\). So \( f \) is differentiable at 0.

Away from 0, \( f'(x) = 2x \sin(1/x) - \cos(1/x) \). Therefore, \( f'(x) \) is not continuous at 0, because \( \lim_{x \to 0} f'(x) \) does not exist, let alone equal \( f'(0) \).

(b) Let \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \) be defined by

\[
g(x, y) = \begin{cases} 
  (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right), & \text{if } (x, y) \neq 0 \\
  0, & \text{if } (x, y) = 0
\end{cases}
\]

Show that \( g \) is differentiable at \((0, 0)\), but \( \frac{\partial g}{\partial x} \) are not continuous at \((0, 0)\).

**Proof.** Again, we claim that \( df_0 = 0 \). Again we use the Squeeze Theorem.

\[
\lim_{h \to 0} \frac{|f(0 + h) - f(0) - 0|}{||h||} = \lim_{h \to 0} \frac{||h||^2 \sin(1/||h||)}{||h||} = \lim_{h \to 0} ||h|| \cdot |\sin(1/||h||)| = 0.
\]

Thus, \( f \) is differentiable at 0.

However,

\[
\frac{\partial f}{\partial x} = 2x \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) - \frac{x}{\sqrt{x^2 + y^2}} \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right),
\]

and

\[
\frac{\partial f}{\partial y} = 2y \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) - \frac{y}{\sqrt{x^2 + y^2}} \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right),
\]

neither of which have a limit at 0. \( \square \)

5. A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is **homogeneous** of degree \( m \) if \( f(tx) = t^m f(x) \) for all \( x \in \mathbb{R}^n \) and all \( t \in \mathbb{R} \). If \( f \) is also differentiable, show that

\[
\sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(x) = mf(x).
\]

**Hint:** Fix \( x \), let \( g(t) = f(tx) \). Compute \( g'(1) \).

**Proof.** Following the hint, we let \( x \in \mathbb{R}^n \), and define \( g(t) = f(tx) \). On the one hand, \( g'(1) = f'(x) \cdot x \). But on the other hand \( g(t) = t^m f(x) \), so \( g'(1) = mf(x) \). Therefore we have

\[
mf(x) = f'(x) \cdot x = \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(x).
\]

\( \square \)