Exercises for Section 1.3

1. Prove the converse of the Generalized Heine-Borel: A compact subset of \( \mathbb{R}^n \) is closed and bounded.

   \textit{Proof.} Let \( A \) be a compact set in \( \mathbb{R}^n \). To see that \( A \) is bounded, consider the open cover
   \[ O = \{ B^n(0, r) : r \in \mathbb{R} \}, \]
   of all open balls in \( \mathbb{R}^n \) centered at the origin. Note that \( O \) is a cover for \( \mathbb{R}^n \), so it is definitely a cover for \( A \). But since \( A \) is compact, then only finitely many are needed: \( \mathcal{U} = \{ B^n(0, r_1), \ldots, B^n(0, r_k) \} \). Let \( M = \max \{ r_1, \ldots, r_k \} \). Then \( A \) is bounded in norm by \( M \).

   Now to show that \( A \) is closed. Let \( p \) be any point outside of \( A \). Now consider the open cover \( \mathcal{F} \) consisting of all sets of the form:
   \[ U_r = \{ x \in \mathbb{R}^n : \|x - p\| > r, r \in \mathbb{R} \}. \]
   Then \( \mathcal{F} \) forms an open cover for \( A \) because it is an open cover for \( \mathbb{R}^n \). But only finitely many are needed, by the compactness of \( A \), say \( U_{r_1}, \ldots, U_{r_k} \). Let \( m = \min \{ r_1, \ldots, r_k \} \). Then \( B^n(p; m/2) \) is an open set containing \( p \) entirely outside of \( A \). Thus \( p \) is in the exterior of \( A \). Therefore, since every point not in \( A \) is in the exterior of \( A \), we see that \( A \) must be closed. \( \square \)

2. If \( U \) is open and \( C \subset U \) is compact, show that there is a compact set \( D \) such that \( C \subset \text{int} \, D \) and \( D \subset U \).

   \textit{Proof.} Because \( C \subset U \), and \( U \) is open, then for each \( x \in C \) there is an open rectangle \( U^x \) with \( x \in U^x \subset U \). Consider the collection of all of these rectangles, one for each and every point in \( C \). For each \( x \), we will “trim down” \( U^x \) to get a smaller open rectangle \( V^x \) in the following way. The \( i \)th side of \( U^x \) should be an interval containing \( x^i \). That is, it looks like \( (x^i - \delta_i, x^i + \epsilon_i) \). Trim off half the distance from \( x^i \) to the boundary on each side, so that the \( i \)th side of \( V^x \) is \( (x^i - \delta_i/2, x^i + \epsilon_i/2) \). Then the closed rectangle \( \text{cl}(V^x) \) is a proper subset of \( U^x \subset U \).

   Now let \( \mathcal{O} \) be the open cover for \( C \) consisting of all the \( V^x \). Since there is one \( V^x \) for each \( x \in C \), this certainly covers \( C \). But \( C \) is compact, so
we only need finitely many, say $V_1, \ldots, V_k$. Let $D = \text{cl}(V_1) \cup \cdots \cup \text{cl}(V_k)$. Since $\text{cl}(V_i)$ is compact, and there are finitely many of them in the union forming $D$, then $D$ must be compact. Moreover, $C \subset (V_1 \cup \cdots \cup V_k) = \text{int}D$ and $D \subset U$. 

3. Show that the intersection of any collection of compact sets is compact. Show that the union of two compact sets is compact.

Proof. The intersection of any collection of closed sets is closed, and the intersection of any collection of bounded sets is bounded. Apply Heine-Borel.

Let $A$ and $B$ be compact sets. Consider any open cover $\mathcal{O}$ for $A \cup B$. Then $\mathcal{O}$ is an open cover for both $A$ and $B$ as well. So finitely many sets $U_1, \ldots, U_k$ in $\mathcal{O}$ will cover $A$, and finitely many more $V_1, \ldots, V_m$ will cover $B$. Together, these $k + m$ sets will cover $A \cup B$. 

4. Show that if $S$ is compact, and $T$ is a closed subset of $S$, then $T$ is compact. Do this in two ways: (1) directly from the definition of compact; and (2) by using Generalized Heine-Borel (see Problem 1).

Proof. (1) Let $\mathcal{O}$ be an open cover for $T$. Since $T$ is closed, then $\mathbb{R}^n - T$ is open. Furthermore, $\mathcal{U} = \mathcal{O} \cup \{\mathbb{R}^n - T\}$ is an open cover for $S$, because $O$ covers all of $T$, and $\mathbb{R}^n - T$ covers everything else. Therefore, because $S$ is compact, finitely many of the sets in $\mathcal{U}$, say $\{U_1, \ldots, U_k, \mathbb{R}^n - T\}$, suffice to cover $S$. But then $\{U_1, \ldots, U_k\}$ must cover $T$, hence $T$ is compact.

(2) All that remains is to show that $T$ is bounded. But $S$ is bounded because it is compact (see Problem 1), so $T$ is clearly bounded as well. Therefore $T$ is compact.

5. Prove that the intersection of connected sets in $\mathbb{R}$ is connected. Show that this is false if “$\mathbb{R}$” is replaced by “$\mathbb{R}^2$.”

Proof. We have shown that connected sets in $\mathbb{R}$ must be intervals. Furthermore, the intersection of intervals is an interval (possibly empty).
Thus, all that remains is to show that all intervals are connected. We will do this in Section ??.

As a counterexample in \( \mathbb{R}^2 \), consider \( A \) to be the unit circle centered at the origin, and \( B \) to be the unit circle centered at \((1,0)\). Each of these are connected sets, but they intersect in two points, a disconnected set. There are many other counterexamples.

6. Prove that if \( E \subset \mathbb{R} \) is connected, then \( \text{int} E \) is also connected. Show that this is false if “\( \mathbb{R} \)” is replaced by “\( \mathbb{R}^2 \).”

Proof. Again, there really isn’t much to prove here. If \( E \subset \mathbb{R} \) is connected, then \( E \) is an interval. Therefore, \( \text{int} E \) is also an interval, hence it is connected.

As a counterexample in \( \mathbb{R}^2 \) consider sets \( A_1 = \{(x,y) \in \mathbb{R}^2 : y \geq |x|\} \) and \( A_2 = \{(x,y) \in \mathbb{R}^2 : y \leq |x|\} \), and let \( A = A_1 \cup A_2 \). Then \( A \) is a connected set, but \( \text{int} A \) has two connected components, namely \( \text{int} A_1 \) and \( \text{int} A_2 \). Other counterexamples abound.

7. Suppose \( A \) is a connected subset of \( E \). Prove that \( A \) lies entirely within one connected component of \( E \).

Proof. Let \( B = \bigcup \{ C \subset E : C \text{ is connected, and } A \subset C \} \). Note that \( A \subset B \) because it is a connected subset of itself.

We now claim that \( B \) is a connected component of \( E \). First, we must show that \( B \) is connected. Suppose \( U \) and \( V \) are disjoint open sets with \( B \subset U \cup V \). Then, because \( A \) is connected, it must lie entirely within one of these two sets, say \( A \subset U \). Moreover, if \( C \) is any connected set with \( A \subset C \), then \( C \) must lie entirely within \( U \) also. So, by the definition of \( B \), every point in \( B \) comes from such a set \( C \), so every point in \( B \) must lie within \( U \). Therefore, \( B \cap V = \emptyset \), so \( B \) is connected.

Now suppose \( D \subset E \) is connected with \( B \subset D \). Then, because \( A \subset B \), then \( A \subset D \). Therefore, \( D \) is a connected subset of \( E \) containing \( A \), hence \( D \subset B \). Thus, \( B = D \), so it must be a component.

8. Suppose that \( E \subset \mathbb{R}^n \) is connected and \( E \subset A \subset \text{cl} E \). Prove that \( A \) is connected.
Proof. If $A = E$ we are done, so we will assume $E$ is a proper subset of $A$. Suppose $A$ can be separated by two non-empty disjoint open sets, $U$ and $V$. Then, because $E$ is connected, it must lie entirely within one of $U$ or $V$. Without loss of generality, suppose $E \subset U$. Notice that because $E \subset A \subset \text{cl}E$, then the only points in $A - E$ are boundary points of $E$. Therefore, if $x \in V \cap A$, then $x \in \partial E$. However, this is a contradiction, because $V$ is an open set containing $x$ which does not intersect $E$, making $x$ an exterior point.

9. (a) Show that $\mathbb{R}^n$ is connected.

Proof. Suppose not; then there are non-empty, non-intersecting open sets $U$ and $V$ with $\mathbb{R}^n = U \cup V$. However, $V$ is open, so $U$ is closed because $U = \mathbb{R}^n - V$ is the complement of an open set. But the only sets that are both open and closed are $\mathbb{R}^n$ and $\emptyset$. Thus we arrive at a contradiction because neither $U$ nor $V$ is empty.

(b) Show that $\mathbb{R} - \{0\}$ is not connected.

Proof. $\mathbb{R} - \{0\}$ can be separated by the sets $(-\infty, 0)$ and $(0, \infty)$.

10. Show that $\mathbb{Q}^n$ is not connected.

Proof. The sets $A = \{q \in \mathbb{Q}^n : q^1 < \pi\}$ and $B = \{q \in \mathbb{Q}^n : q^1 > \pi\}$ are two disjoint open sets that serve to separate $\mathbb{Q}^n$. Other examples are equally valid.