Math 3210-1    HW 6    Solutions

Limits of Functions

20.2 \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 0 \). \( \lim_{x \to +\infty} f(x) = +\infty \). \( \lim_{x \to -\infty} f(x) = -\infty \).

20.4 \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 0 \). \( \lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 1 \).

20.8 Note that \(-|x| \leq x \sin \left( \frac{1}{x} \right) \leq |x|\). Thus, we may apply the Squeeze Theorem, and the result follows.

Continuity

17.6 Rational functions are quotients of polynomials. Polynomials are continuous by Exercise 17.5, and fractions of continuous functions are continuous away from zeros of the denominator. Since the domain of a rational function does not include zeros of the denominator, we can apply Theorem 17.4.

17.7 (a) This follows directly from Exercise 17.5

(b) By definition of \(|x|\), it is continuous away from \(x = 0\), using part (a). To see that it is continuous at \(x = 0\) simply note that \(\lim_{x \to 0} |x| = 0\).

(c) If \(f\) is continuous, then \(|f|\) is continuous since it is the composition of two continuous functions (namely \(f\) and the absolute value function).

17.13 (b) Notice that \(-|x| \leq h(x) \leq |x|\) for all \(x \in \mathbb{R}\). Thus, by the Squeeze Theorem, \(\lim_{x \to 0} h(x) = 0 = h(0)\). Therefore, \(h\) is continuous at 0.

To see that \(h\) is discontinuous at \(x \neq 0\), simply repeat the sequence argument from part (a).

Properties of Continuous Functions

18.6 We wish to prove that \(x = \cos x\) for some \(x \in (0, \frac{\pi}{2})\). Toward this end, let \(f(x) = \cos x - x\). Then this is equivalent to showing that \(f\) has a zero on this interval. Note that \(f\) is continuous, and that \(f(0) = 1\), while \(f(\pi/2) = -\pi/2\). Thus, we may apply the Intermediate Value Theorem.

18.8 Note that \(f(a)\) and \(f(b)\) must have opposite signs, and neither is zero. Otherwise, \(f(a)f(b)\) would be greater than or equal to zero. We may assume that \(f(a) < 0\) and \(f(b) > 0\). Thus, since \(f\) is continuous, we may apply the Intermediate Value Theorem. Thus there exists some \(x \in (a, b)\) such that \(f(x) = 0\).