7.2 (a) \((s_n)\) converges to 0.

(b) \((b_n)\) converges to \(\frac{3}{4}\)

(c) \((c_n)\) converges to 0.

(d) DOES NOT CONVERGE.

7.4 (a) Let \(x_n = \frac{\sqrt{n}}{n}\), then \(x_n\) is irrational \(\forall n \in \mathbb{N}\), but \((x_n) \to 0\) which is rational.

(b) Let \(r_n = (1 + \frac{1}{n})^n\), then \(r_n\) is rational \(\forall n \in \mathbb{N}\), and \((r_n) \to e\).

8.4 Let \((t_n)\) be a bounded sequence, and let \((s_n)\) converge to zero. Prove that \((s_n \cdot t_n) \to 0\).

Proof: Since \((t_n)\) is bounded, \(\exists M > 0\) such that \(|t_n| \leq M\) for all \(n \in \mathbb{N}\).

Let \(\varepsilon > 0\). Then \(\exists N \in \mathbb{N}\) such that \(n > N\) implies \(|s_n| < \frac{\varepsilon}{M}\).

Thus \(n > N\) implies

\[|t_n s_n| = |t_n||s_n| \leq M|s_n| < \varepsilon.\]
9.2 \( (x_n) \rightarrow 3 \), \( (y_n) \rightarrow 7 \) and all \( y_n \neq 0 \).

(a) \( (x_n + y_n) \rightarrow 10 \).

(b) \[ \left( \frac{3y_n - x_n}{y_n^2} \right) \rightarrow \frac{18}{49} \.

9.8.

(a) Verify \( 1 + a + a^2 + \ldots + a^n = \frac{1-a^{n+1}}{1-a} \), for \( a \neq 1 \).

**Proof:** Let \( S = 1 + a + \ldots + a^n \),

then \( S - a \cdot S = 1-a^{n+1} \).

Solve for \( S \),
\[ S = \frac{1-a^{n+1}}{1-a} \] .

(b) \( \lim_{n \to \infty} (1 + a + \ldots + a^n) = \lim_{n \to \infty} \frac{1-a^{n+1}}{1-a} \).

when \( |a| < 1 \) this converges to \( \left[ \frac{1}{1-a} \right] \).

(c) Using (b) we get \( \left[ \frac{1}{1-a} \right] \) (a = \( \frac{1}{3} \)).

(d) This sequence diverges to +\( \infty \).
10.4 In the framework of rational numbers, the supremum of a set could possibly not exist.

10.8 Let \((s_n)\) be nondecreasing.

Define \(\bar{r}_n = \frac{s_1 + \cdots + s_n}{n}\).

Prove \(\bar{r}_n\) is nondecreasing.

**Proof:** We need to show \(\bar{r}_{n+1} \geq \bar{r}_n\).

Equivalently, we can show

\[(n+1)(s_1 + \cdots + s_n) \leq n(s_1 + \cdots + s_{n+1}). \quad \text{(Why?)}\]

Consider the left-hand side:

\[(n+1)(s_1 + \cdots + s_n) = n(s_1 + \cdots + s_n) + (s_1 + \cdots + s_n).

But \(s_k \leq s_{n+1}\) for \(1 \leq k \leq n+1\), since \((s_n)\) is nondecreasing.

Thus \((s_1 + \cdots + s_n) \leq s_{n+1} + s_{n+1} + \cdots + s_{n+1} = n(s_{n+1})\).

Thus,

\[
(n+1)(s_1 + \cdots + s_n) = n(s_1 + \cdots + s_n) + (s_1 + \cdots + s_n) \\
\leq n(s_1 + \cdots + s_n) + n s_{n+1} \\
= n(s_1 + \cdots + s_{n+1}).
\]
\[ S_1 = 1 \]
\[ S_{n+1} = \frac{1}{3} (S_n + 1) \quad \text{for } n \geq 1 \]

(a) \[ S_2 = \frac{2}{3}, \quad S_3 = \frac{1}{3} \left( \frac{2}{3} + 1 \right) = \frac{5}{9}, \quad S_4 = \frac{1}{3} \left( \frac{5}{9} + 1 \right) = \frac{14}{27} . \]

(b) \underline{Claim: } \( S_n > \frac{1}{2} \) for all \( n \).

\underline{Proof: } \( S_1 > \frac{1}{2} \). Now suppose \( S_k > \frac{1}{2} \).

\[ S_{k+1} = \frac{1}{3} (S_k + 1) > \frac{1}{3} \left( \frac{1}{2} + 1 \right) = \frac{1}{2} . \]

Thus, the result holds by induction.

(c) \underline{Claim: } \( S_{n+1} \leq S_n \) for all \( n \).

\[ S_2 = \frac{2}{3} < 1 = S_1 . \]

Suppose \( S_{k+1} \leq S_k \). Look at \( S_{k+2} \).

\[ S_{k+2} = \frac{1}{3} (S_{k+1} + 1) \leq \frac{1}{3} (S_k + 1) = S_{k+1} . \]

Thus, by induction, \( (S_n) \) is non-increasing.

(d) \[ \lim_{n \to \infty} (S_n) \text{ exists by the Monotone Convergence Theorem} . \]

To calculate \( \lim_{n \to \infty} S_n \), we let \( S \) be this limit. Then, since \( \lim_{n \to \infty} S_{n+1} = \lim_{n \to \infty} S_n \), we see that

\[ \frac{1}{3} (S + 1) = S \]

must hold. Therefore \( S = \frac{1}{2} \), and \( (S_n) \to \frac{1}{2} \).