1. (a) If there exist violets which are not blue, then there exist roses which are not red.
(b) If \( G \) is not normal then \( G \) is regular.
(c) If \( K \) is not compact, then \( K \) is not closed or not bounded.

2. (a) If all violets are blue, then all roses are red.
(b) If \( G \) is normal, then \( G \) is not regular.
(c) If \( K \) is compact, then \( K \) is closed and bounded.

3. (a) \( x = -3 \)
(b) \( n = 41 \), then \( n^2 + n + 41 = 41(41 + 1 + 1) \); i.e., not prime.
(c) \( \frac{1}{2} \sqrt{2} \)
(d) Let \( n = 101 \), or let \( n = 2^{24,036,583} - 1 \), the 41st known Mersenne Prime.  
See http://www.mersenne.org/prime.htm
(e) Let \( p = 2 \).
(f) Let \( n = 1, 3, 5, \ldots \)
(g) \( x = 0 \)
(h) \( x = -1 \).

4. The contrapositive is: If \( f(x_1) = f(x_2) \) then \( x_1 = x_2 \).

proof: Suppose \( f(x_1) = f(x_2) \). Then 
\[
3x_1 - 5 = 3x_2 - 5
\]
add 5: \( 3x_1 = 3x_2 \)
divide: \( x_1 = x_2 \).

5. The contrapositive is: If \( n \) is odd, then \( n^2 \) is odd.

proof: If \( n \) is odd, then there exists \( k \) so that \( n = 2k + 1 \).
Then \( n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \).
Therefore \( n^2 \) is odd.
6. \((A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)\).

Proof: we need to show \((A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)\) and 
\[(A \setminus B) \cup (B \setminus A) \supseteq (A \cup B) \setminus (A \cap B)\].

We will show \(\subseteq\) first:
Let \(x \in (A \setminus B) \cup (B \setminus A)\). Then \(x \in (A \setminus B)\) or \(x \in (B \setminus A)\).
If \(x \in A \setminus B\) then \(x \in A\) and \(x \notin B\). Thus, \(x \in A \cup B\) and \(x \notin A \cap B\).
Thus \(x \in (A \cup B) \setminus (A \cap B)\). Likewise if \(x \in (B \setminus A)\) then
\(x \in B\), \(x \notin A\). Thus \(x \in A \cup B\) and \(x \notin A \cap B\). Hence we conclude \(x \in (A \cup B) \setminus (A \cap B)\).

Now to show \(\supseteq\):
Let \(y \in (A \cup B) \setminus (A \cap B)\). Thus \(y \in A\) or \(y \in B\), and \(y \notin A \cap B\).
If \(y \notin A \cap B\) then \(y \notin A\) or \(y \notin B\). But we know \(y\) is in one of them.
If \(y \notin A \cap B\) then \(y \notin A\) or \(y \notin B\). But we know \(y \in A\) or \(y \in B\).
Therefore we conclude that either \(y \in A\) and \(y \notin B\) (i.e. \(y \in A \setminus B\)),
or \(y \in B\) and \(y \notin A\) (i.e. \(y \in B \setminus A\)). Thus, \(y \in (A \setminus B) \cup (B \setminus A)\).

7. \(A \cap B = A \setminus (A \setminus B)\).

Proof: Again, we will prove \(\subseteq\) first.
Let \(x \in (A \cap B)\). Then \(x \in A\) and \(x \in B\). Since \(x \in B\), \(x \notin A \setminus B\).
Therefore, \(x \in A \setminus (A \setminus B)\).

Now we show \(\supseteq\):
Let \(y \in A \setminus (A \setminus B)\). Then \(y \in A\) and \(y \notin (A \setminus B)\). Since \(y \notin (A \setminus B)\) then either \(y \in A\) or \(y \in B\). But we know \(y \in A\). Thus \(y \in B\).
Since \(y \in A\) and \(y \in B\), \(y \in A \cap B\).
8. \( A \times B = B \times A \).

This is FALSE. The cartesian product is ordered:

\[
A \times B = \{(a,b) : a \in A, b \in B\}.
\]

Counterexample:

Let \( A = \mathbb{Z} \), \( B = \mathbb{R} \).

Then \( \mathbb{Z} \times \mathbb{R} \) is the set of ordered pairs whose first entry is an integer and whose second is real, while \( \mathbb{R} \times \mathbb{Z} \) has reals in the first slot and integers in the second.

i.e. \((\pi, 1) \in \mathbb{R} \times \mathbb{Z} \),

\((\pi, 1) \notin \mathbb{Z} \times \mathbb{R} \).

9. (a) \((A \cup B) \times C = (A \times C) \cup (B \times C)\).

**Proof:** First we show \( \subseteq \)

Let \((x,c) \in (A \cup B) \times C \). Then \( x \in A \cup B \), \( c \in C \). Then either \( x \in A \) or \( x \in B \). If \( x \in A \), then \((x,c) \in A \times C \). If \( x \in B \), then \((x,c) \in B \times C \). So \((x,c) \in A \times C \) or \((x,c) \in B \times C \). Thus \((x,c) \in (A \times C) \cup (B \times C) \).

Now to show \( \supseteq \).

Let \((y,d) \in (A \times C) \cup (B \times C) \). Then either \((y,d) \in A \times C \) or \((y,d) \in B \times C \). Either way, \( d \in C \). Furthermore, either \( y \in A \) or \( y \in B \). Thus \( y \in A \cup B \). So \((y,d) \in (A \cup B) \times C \).
9. (b) \((A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)\).

Proof: "\subseteq":
Let \((x, y) \in (A \times B) \cap (C \times D)\). Then \((x, y) \in A \times B\) and \((x, y) \in C \times D\). Thus \(x \in A\), \(y \in B\) and \(x \in C\), \(y \in D\). Thus \(x \in A \cap C\) and \(y \in B \cap D\). Thus \((x, y) \in (A \cap C) \times (B \cap D)\).

"\supseteq":
Let \((x, y) \in (A \cap C) \times (B \cap D)\). Then \(x \in A \cap C\), \(y \in B \cap D\). So \(x \in A\), \(y \in B\) and \(x \in C\), \(y \in D\). Thus \((x, y) \in A \times B\) and \((x, y) \in C \times D\). Thus, \((x, y) \in (A \times B) \cap (C \times D)\). \(\qed\)

(c) \((A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)\).

This is FALSE!

Consider the following pictures:

\((A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)\).
10. (a) RT
(b) R
(c) RST
(d) S
(e) ST
(f) RS

11. (a) \( x \, R \, y \iff x < y + 2 \)
(b) \( x \, R \, y \iff x \neq y \)
(c) \( A \, R \, B \iff A \subseteq B \) for sets, \( A, B \).
(d) Define \( R \) on lines in the plane where 
\( l \, R \, k \iff \) the lines \( l \) and \( k \) have 
at least one point in common.
(e) The relation \( \leq \) defined on \( \mathbb{R} \).
(f)

12. \( (a, b) \, R \, (c, d) \iff a = c \).

Reflexive: Certainly \( (a, b) \, R \, (a, b) \).
Symmetric: If \( (a, b) \, R \, (c, d) \) then \( a = c \), so \( c = a \). Thus \( (c, d) \, R \, (a, b) \).
Transitive: If \( (a, b) \, R \, (c, d) \) and \( (c, d) \, R \, (e, f) \) then \( a = c \) and \( c = e \) 
Thus, \( a = e \) and \( (a, b) \, R \, (e, f) \).

The class \( E_{(a, b)} \) is all ordered pairs with first entry \( a \).

\[ E_{(a, b)} = \{ (a, x) : x \in \mathbb{R} \} \]

Geometrically, \( E_{(a, b)} \) is the vertical line through \( (a, b) \).
13. \( S = \mathbb{Z}, \quad R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m-n \text{ is even}\} \).

Reflexive: \( \forall m \in \mathbb{Z}, \quad m-m = 0 \) thus \( mRm \).

Symmetric: If \( mRn \) then \( m-n \) is even, thus, \( n-m \) is even. Hence \( nRm \).

Transitive: If \( mRn \) and \( nRp \) then \( m-n \) is even and \( n-p \) is even. Thus \( m-p \) is even. Therefore \( mRp \).

\( E_5 \) is the set of odd integers.

There are only two equivalence classes. Namely, \( E_5 \) and \( E_{-364} \).

Functions

14. (a) \( f(x) = x^2 + 1 \)

\[ \text{range}(f) = [1, \infty) \]

(b) \( f(x) = (x+3)^2 - 5 \)

\[ \text{range}(f) = [-5, \infty) \]

(c) \( f(x) = x^2 + 4x + 1 \)

\[ f'(x) = 2x + 4 \quad f'(x) = 0 \text{ at } x = -2 \]

So \( f(-2) = -3 \) is the minimum.

\[ \text{range}(f) = [-3, \infty) \]

(d) \( f(x) = 2 \cos 3x \)

\[ \text{range}(f) = [-2, 2] \]
15. (a) \( f \) is not injective since there are many (infinitely many) circles with the same area. \( f \) is, however, surjective since we can find a circle with any given area in \([0, \infty)\).

(b) Now, since we have prescribed the center point for the circles, \( g \) is both injective and surjective.

16. It may seem there is nothing to prove, but there is: We must show that these functions two functions \( (k = h \circ (g \circ f) \text{ and } l = (h \circ g) \circ f) \) take each point \( x \in A \) to the same place in \( D \).

Proof: We need to show \( k(x) = l(x) \) for all \( x \in A \).

Let \( x \in A \). Define \( y = f(x) \) in \( B \), \( z = g(y) \) in \( C \), \( w = h(z) \) in \( D \).

Now \( (g \circ f)(x) = g(f(x)) = g(y) = z \), and
\[
(h \circ g)(y) = h(g(y)) = h(z) = w.
\]

Thus,
\[
k(x) = h \circ (g \circ f)(x) = h((g \circ f)(x)) = h(z) = w
\]
and
\[
l(x) = (h \circ g) \circ f(x) = (h \circ g)(f(x)) = (h \circ g)(y) = w.
\]

Therefore, \( \forall x \in A, k(x) = l(x) \). Thus, \( h \circ (g \circ f) = (h \circ g) \circ f \).

17. \( f: \mathbb{N} \to \mathbb{N} \)

(a) surjective, not injective: \( f(n) = \lceil n/2 \rceil \) (which we read as the "ceiling" of \( n/2 \)).

\[f(n) = \lceil n/2 \rceil \] means, divide \( n \) by \( 2 \) and round up when \( n \) is odd,

\[
\begin{align*}
1 & \rightarrow 1 \\
2 & \rightarrow 1 \\
3 & \rightarrow 2 \\
4 & \rightarrow 2 \\
5 & \rightarrow 3 \\
6 & \rightarrow 3 \\
7 & \rightarrow 4 \\
& \vdots
\end{align*}
\]
(b) injective, not surjective: \( f(n) = p_n \), the \( n \)th prime.

\[
\begin{align*}
1 & \mapsto 2 \\
2 & \mapsto 3 \\
3 & \mapsto 5 \\
4 & \mapsto 7 \\
5 & \mapsto 11 \\
6 & \mapsto 13 \\
7 & \mapsto 17 \\
& \quad \ddots
\end{align*}
\]

(c) neither surjective nor injective: \( f(n) = 31 \), the constant function.

d) bijective: \( f(n) = n \), the identity function, is the most obvious. Here is another:

\[
\begin{align*}
1 & \mapsto 2 \\
2 & \mapsto 1 \\
3 & \mapsto 5 \\
4 & \mapsto 3 \\
5 & \mapsto 4 \\
6 & \mapsto 9 \\
7 & \mapsto 6 \\
8 & \mapsto 7 \\
9 & \mapsto 8 \\
& \quad \ddots
\end{align*}
\]

Do you see the pattern? Note there need not be a pattern so long as each number is "hit" exactly once.
18. \( f: A \to B, \ C \subseteq A, \ D \subseteq B. \)

(a) \( f(C) \subseteq D \text{ iff } C \subseteq f^{-1}(D). \)

Proof: There are two implications to prove. We will first prove \( \Rightarrow \).

Suppose \( f(C) \subseteq D. \) Therefore, if \( x \in C \) then \( f(x) \in D. \)

But \( f^{-1}(D) = \{ x \in A : f(x) \in D \}. \) Thus, \( C \subseteq f^{-1}(D). \)

Now to show \( \Leftarrow. \)

Suppose \( C \subseteq f^{-1}(D). \) That is, if \( x \in C \) then \( f(x) \in D. \)

Let \( y \in f(C). \) Then there exists \( z \in C \) so that \( y = f(z). \)

But \( z \in C \) implies \( f(z) \in D. \) Thus \( y \in D, \) and we conclude \( f(C) \subseteq D. \)

(b) For \( f(C) = D \text{ iff } C = f^{-1}(D) \) we require that \( D \) be in the image of \( f, \) that is \( D \subseteq f(A). \)
19. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) via \( f(x) = e^x \) and 
\( g : \mathbb{R} \rightarrow \mathbb{R} \) via \( g(x) = x^2 \)

Then \( (g \circ f)(x) = e^{x^2} \) is injective, but \( g \) is not injective.

20. \( g : A \rightarrow C \), \( h : B \leftrightarrow C \) is bijective, then \( \exists f : A \rightarrow B \) so that \( g = h \circ f \).

proof: This is the picture we want.

\[
\begin{array}{c}
A \xrightarrow{g} C \\
| \\
\downarrow f \quad \downarrow h \\
| \\
\text{so that} B \\
g = h \circ f.
\end{array}
\]

Since \( h \) is bijective, for every \( y \in C \) there is exactly one \( z \in B \) so that \( h(z) = y \). Therefore, we can declare \( h^{-1}(y) = z \). (we say the inverse is "well-defined").

Now we will define the function \( f : A \rightarrow B \).
Let \( x \in A \). Then \( g(x) \in C \). By what we said above, there exists a unique \( h^{-1}(g(x)) \in B \) so that \( h(h^{-1}(g(x))) = g(x) \).
Let \( f(x) = h^{-1}(g(x)) \). Then we conclude that \( g = h \circ f \).