Homework 13: Review – old quals plus ϵ

I collected some old qual problems. I divided them into two parts. Part A is "Cauchy and friends" and roughly corresponds to the material in 4200. Part B is the more advanced part of 6220. I also added Part C to cover some topics I didn't find on old quals.

You should also be ready to state the following named theorems: Cauchy-Riemann, Goursat, Morera, Cauchy integral formula, Liouville, Schwarz reflection, Runge, Argument principle, Open Mapping, Rouché, Schwarz lemma, Montel, Picard – little and great, Riemann Mapping theorem, Schwarz-Christoffel, Mittag-Leffler, Weierstrass Product theorem, Uniformization.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ be the upper half plane and let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk.

Part A.

1. Let f be an entire function such that $|f(z)| \leq K|z|^n$ where K is a positive real constant and n is a positive integer. Show that f is a polynomial of degree $\leq n$.

2. Calculate the integral
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4} dx$$

3. Calculate the integral
$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$$

4. Let $a > 1$ be arbitrary. Show that the equation $a - z - e^{-z} = 0$ has exactly one solution in the half-plane $\{z \mid Re(z) > 0\}$, and moreover, this solution is real.

5. Does there exists a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ so that
$$f\left(\frac{1}{n}\right) f\left(\frac{1}{n+1}\right) = \frac{1}{n}$$
for $n \in \{2, 3, ...\}$?

6. Let $\mathcal{F}$ be a family of holomorphic functions on $\mathbb{D}$ so that there exists $C > 0$ with $|f(z)| \leq C$ for all $z \in \mathbb{D}$.

   (a) If $K \subset \mathbb{D}$ is compact then $\mathcal{F}$ is uniformly Lipshitz on K. That is, there exists $\lambda > 0$ so that for any $f \in \mathcal{F}$ and $z, w \in K$ we have $|f(z) - f(w)| \leq \lambda |z - w|$.

   (b) Show that $\mathcal{F}$ is not necessarily uniformly Lipschitz on $\mathbb{D}$.

7. State and prove Morera's Theorem.

8. Using Rouché's theorem find the number of zeros of the polynomial $2z^5 - z^3 + 3z^2 - z + 8$ in the region $\{z \mid |z| > 1\}$.

9. Describe all entire functions f which satisfy the property:

$$\lim_{z \to \infty} \frac{1}{f(z)} = 0$$

10. Evaluate

$$\int_C \frac{\sin z}{z^6} dz$$

where C is the positively oriented unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.

Part B.

11. (a) Find a biholomorphic map from $\mathbb{H}$ to $\mathbb{D}$ that takes i to 0.

    (b) Let $f : \mathbb{H} \to \mathbb{H}$ be a holomorphic map with $f(i) = i$. Show that $|f'(i)| \leq 1$ and that if $|f'(i)| = 1$ then f is of the form $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$.

12. Let f be an entire function such that for all $x \in \mathbb{R}$, $f(ix)$ and $f(1+ix)$ are in $\mathbb{R}$. Show that f is periodic with period 2. That is, show that $f(z) = f(z+2)$ for all $z \in \mathbb{C}$.

13. Let f be a function holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Assume that $|f(z)| = 1$ whenever $|z| = 1$. Show that f can be extended to a meromorphic function on all of $\mathbb{C}$, with at most finitely many poles. Further, prove that f is the restriction of a rational function.

14. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and zero at points $z_1, \cdots, z_n \in \mathbb{D}$ counted with multiplicity. Show that

$$|f(z)| \leq |\psi_{z_1}(z)| \cdots |\psi_{z_n}(z)|$$

in $\mathbb{D}$ where $\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$.

15. Let Ω be a simply connected domain in $\mathbb{C}$. Show that if a holomorphic function $f : \Omega \to \mathbb{C}$ has finitely many zeros, all of even order, then f has a holomorphic square root in Ω, i.e. there is a holomorphic function $g : \Omega \to \mathbb{C}$ so that $f(z) = g(z)^2$.

16. let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Show that

$$|f(z) - f(0)| \leq |z(1 - \overline{f(0)}f(z))|$$

for all $z \in \mathbb{D}$.

17. Let $\wp(z)$ be the Weierstrass $\wp$-function with periods ω_1, ω_2. Show that any even elliptic function $f(z)$ with the same periods can be expressed in the form

$$f(z) = C \prod_{k=1}^{n} \frac{\wp(z) - a_k}{\wp(z) - b_k}$$

for some constants $C, a_k, b_k \in \mathbb{C}$, provided that 0 is neither a zero nor a pole. What is the corresponding form if $f(z)$ either vanishes at 0 or has a pole at 0?

Part C.

18. Compute the hyperbolic distance in the upper half-plane $\mathbb{H}$ between xi and $1 + xi$, for $x > 0$.

19. Let $A_R = \{z \in \mathbb{C} \mid 1 < |z| < R\}$ for $R > 1$.

    (a) Show that if A_R and $A_{R'}$ are biholomorphic, then $R = R'$.

    (b) Show that every holomorphic map $\mathbb{C} \setminus \{0\} \to A_R$ is constant, for any $R > 1$.

20. Define $\log z$ on $\mathbb{C} \setminus \{z \mid Re(z) = 0, Im(z) \leq 0\}$ by

$$\log re^{i\theta} = \log r + i\theta$$

for $\theta \in (-\pi/2, 3\pi/2)$ and define $z^{-2/3} = e^{(-2\log z)/3}$. Describe the image of the function

$$f(z) = \int_0^z \xi^{-2/3}(\xi - 1)^{-2/3} d\xi$$

defined on $\mathbb{H}$. You don't have to give the exact coordinates, but you should find the geometric shape.