Homework 9: Hyperbolic geometry

Notation: $\mathbb{D} = \{z| \mid |z| < 1\}$, the Poincaré disk model, and $\mathbb{H} = \{z \mid Im(z) > 0\}$, the upper half-plane model of the hyperbolic plane $\mathbb{H}^2$. Also, S^1_∞ is the circle at infinity, represented by $\partial\mathbb{D} = \{|z| = 1\}$ or by $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$.

Synthetic geometry.

1. In class we have seen that given a geodesic ℓ and a point x there is a unique geodesic through x perpendicular to ℓ. Show that if ℓ, ℓ' are two disjoint geodesics whose endpoints on S^1_∞ are all distinct, then there is a unique geodesic perpendicular to both. Show also that if ℓ, ℓ' have a common endpoint (we say they are asymptotic) but are distinct, then there is no common perpendicular.

Distance computations.

Very useful distances to remember are:

- In $\mathbb{H}$: $d(ai, bi) = \log(b/a)$ if $0 < a < b$.
- In $\mathbb{D}$: $d(0, r) = \log \frac{1+r}{1-r}$, and of course $d(0, z) = d(0, |z|)$.

2. Consider the points i and $x + i$ in $\mathbb{H}$, with $x > 0$.

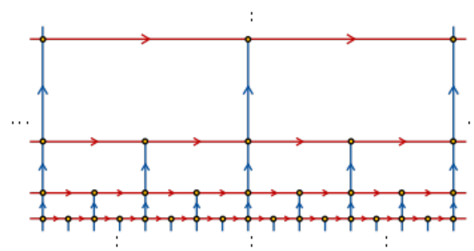
   (i) Compute the hyperbolic distance between these two points and observe that for $x \to \infty$ it is approximately $2 \log x$. Hint: There are many ways to do this, but I think the easiest is to map $x + i$ to $\mathbb{D}$ by a Möbius transformation $\mathbb{H} \to \mathbb{D}$ that maps i to 0 and then take $\log \frac{1+r}{1-r}$ where r is the norm of the image point.

   (ii) Compute the hyperbolic length of the path $t \mapsto t + i$ for $t \in [0, x]$ and observe that it is exponentially longer for large x.

   (iii) Compute the length of the path $[i, xi] \cup [xi, x + xi] \cup [x + xi, x + i]$ and show that it is comparable to the hyperbolic distance between the endpoints (up to a uniformly bounded error).

This comes up in "real life". If you are on google maps observing the math department at the U and you want to look at the Eiffel Tower, you would first zoom out, then go over to France, and then zoom in. The above is a model of this with zoomed in level represented by the horocycle $Im(z) = 1$. For a 2-dimensional model one would look at the hyperbolic 3-space $\mathbb{H}^3$.

There is also a combinatorial model for hyperbolic plane. It is pictured below, with all edges having length 1. It is good for rough calculations. For example, check the above calculations on the model.



3. The goal of this exercise is to show that when $a, b \in \mathbb{D}$ then the hyperbolic distance

$$d(a,b) = \log \frac{1 + \frac{|a-b|}{|1-a\bar{b}|}}{1 - \frac{|a-b|}{|1-a\bar{b}|}}$$

It is more complicated than our cross-ratio formula, but the advantage is that it doesn't involve the endpoints of the geodesic through a and b.

(i) If $z_0 \in \mathbb{D}, \theta \in \mathbb{R}$ then the transformation

$$f : z \mapsto e^{i\theta} \frac{z - z_0}{1 - z\bar{z}_0}$$

is an automorphism of $\mathbb{D}$.

(ii) Every automorphism of $\mathbb{D}$ has the form from (i).

(iii) Denote

$$\rho(a,b) = \left| \frac{a-b}{1-a\bar{b}} \right|$$

for $a, b \in \mathbb{D}$. Note that $\rho(a,b) = \rho(b,a)$. The transformation f from (i) sends $z_0 \mapsto 0$. Show that

$$\rho(z, z_0) = \rho(f(z), 0)$$

(iv) If g is any automorphism of $\mathbb{D}$, show that $\rho(a,b) = \rho(g(a), g(b))$.

(v) Calculate directly that $\rho(0,r) = r$.

(vi) Deduce the distance formula.

4. In $\mathbb{H}$ consider the geodesic $Re(z) = 0$ and two straight lines $L_\pm$ through the origin with slopes $\pm k$ for some $k > 0$. Show that the distance from any point of $L_\pm$ to $Re(z) = 0$ is some constant $C(k)$. You don't have to compute anything, just use the isometry group. These two lines are called "equidistant lines". Also show that if $|Re(w)/Im(w)| < 1/k$ then the distance between w and $Re(z) = 0$ is less than $C(k)$ and if $|Re(w)/Im(w)| > 1/k$ then the distance is $> C(k)$. Thus a metric neighborhood of $Re(z) = 0$ is a sector. Sketch a metric neighborhood of a geodesic in $\mathbb{H}$ represented by a semi-circle, and a metric neighborhood of a geodesic in $\mathbb{D}$. The name on the street for these is a "banana neighborhood".

Isometries.

5. Suppose $f, g \in Isom_+(\mathbb{H}^2)$ are two commuting isometries and neither is identity. Show that one of the following holds:

   (i) both f, g are elliptic and fix the same point of $\mathbb{H}^2$, or

   (ii) both f, g are parabolic and fix the same point on S^1_∞, or

   (iii) both f, g are hyperbolic and have the same axis.

   As an example, $z \mapsto z + 1$ and $z \mapsto 2z$ do not commute, and in fact generate a solvable group called the Baumslag-Solitar group $BS(2,1)$. Conclude that the rank 2 free abelian group $\mathbb{Z}^2$ doesn't act discretely and faithfully by orientation preserving isometries on $\mathbb{H}^2$.

6. Here we view $Isom_+(\mathbb{H}^2) = PSL_2(\mathbb{R})$ as acting on $\mathbb{H}^2 \cup S^1_\infty$. This is a Lie group.

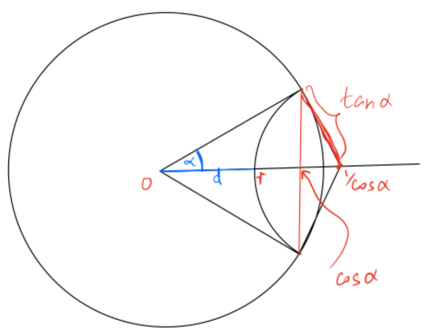
   (i) Show that the stabilizer of a point in $\mathbb{H}^2$ is a compact subgroup isomorphic to the circle $SO(2)$. (This is in fact a maximal compact subgroup of $PSL_2(\mathbb{R})$.)

   (ii) Show that the stabilizer of a point in S^1_∞ is a noncompact solvable Lie group.

Hyperbolic Trigonometry.

This is what a 1060 final in a hyperbolic universe might look like.

7. Consider a geodesic triangle with one ideal vertex, one right angle, the finite side of hyperbolic length d, and the third angle α. Prove the Lobachevsky formula
$$\sin\alpha = \frac{1}{\cosh d}$$
As a guide, use the following figure. First show that $r = \frac{1}{\cos\alpha} - \tan\alpha$.



Thus if you lived in hyperbolic space, a straight line would appear to you at a visual angle $< \pi$ and moreover, from this visual angle you could compute the distance to the line!

8. There are all kinds of trig formulas, I'll post some on the web page. Consider the following three:

- (Pythagorean theorem) If a, b, c are the sides of a right triangle with c opposite to the right angle, then
$$\cosh c = \cosh a \cosh b$$

- (Law of Sines) In any triangle with sides a, b, c and opposite angles α, β, γ
$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}$$

- (Law of Cosines)

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$$

(this generalizes the Pythagorean theorem)

I am not asking you to prove these, but consider the following thought experiment. Rescale the metric on $\mathbb{H}^2$ by larger and larger number. This amounts to "zooming in". In the limit you will get a metric space isometric to the Euclidean plane. By taking a sequence of suitably smaller and smaller triangles in $\mathbb{H}^2$ and rescaling, these triangles will converge to a Euclidean triangle. So one should be able to deduce Euclidean trig formulas from the hyperbolic ones! Here is a concrete question: Fix $a, b > 0$ and consider a hyperbolic right triangle with sides ta, tb and hypotenuse $tc(t)$, for $t > 0$. Show that

$$(c(t)^2 - a^2 - b^2)/t^2 \to 0$$

as $t \to 0$. So in the limit $a, b, c(0)$ satisfy the Euclidean Pythagorean theorem.

Do the same with the Law of Sines and deduce the Euclidean Law of Sines.

Gromov's δ-hyperbolicity.

Let $\delta \geq 0$. Unlike in calculus, this δ doesn't have to be small. A triangle with vertices A, B, C and sides AB, AC, BC is said to be δ-thin if for every point x on one of the sides, there is a point y on another side with $d(x, y) \leq \delta$. Vertices belong to two sides, so you may as well assume x is not a vertex, otherwise $y = x$ works. If you take a large geodesic triangle in Euclidean plane, it will be δ-thin only for large δ. A metric space is said to be *δ-hyperbolic* if there is some $\delta \geq 0$ such that every geodesic triangle is δ-thin. One also says a metric space is Gromov hyperbolic if it is δ-hyperbolic for some δ. This definition captures the essential features of negative curvature. So $\mathbb{R}^2$ with the usual metric is not Gromov hyperbolic.

9. Show that $\mathbb{H}^2$ is Gromov hyperbolic. Hint: The "worst" triangle is an ideal triangle, and you should show that it is δ-thin for $\delta = \log(1+\sqrt{2})$.

10. Use δ-thinness of geodesic triangles to prove that every geodesic quadrilateral is 2δ-thin: every point on one of the sides is within distance 2δ of another side.

11. Use the previous problem to show the following. Let ℓ be a geodesic in $\mathbb{H}^2$, $x, y \in \mathbb{H}^2$ two points and x', y' their nearest point (or orthogonal) projections to ℓ. Show that there is a constant C such that if $d(x', y') > 4\delta$ (the hyperbolicity constant) then the length of

$$[x, x'] \cup [x', y'] \cup [y', y]$$

is less than $d(x, y) + C$. So if x', y' are sufficiently far apart, to go from x to y you may as well (up to uniformly bounded detour) go straight to ℓ, then follow ℓ to y' and then head straight to y.

12. Gromov hyperbolicity is equivalent to the statement that the radius of an inscribed circle in every geodesic triangle is uniformly bounded. Calculate the radius of the inscribed circle in an ideal triangle. Answer: $(\log 3)/2$.