Homework 8: Möbius transformations, Riemann surfaces

Notation: $\mathbb{D} = \{z| \mid |z| < 1\}$ and $\mathbb{H} = \{z \mid Im(z) > 0\}$.

Möbius transformations.

1. Determine which of the following transformations are elliptic, parabolic, loxodromic. Recall that I don't distinguish between "hyperbolic" and "loxodromic" (if you are looking at Ahlfors).

   (i) $z \mapsto z/(2z-1)$.

   (ii) $z \mapsto 2z/(3z-1)$.

   (iii) $z \mapsto (3z-4)/(z-1)$.

   (iv) $z \mapsto z/(2-z)$.

2. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. If $f(0) = a$ show that f has no zeros in $\{|z| < |a|\}$. Hint: Find an automorphism of $\mathbb{D}$ that interchanges 0 and a and apply the Schwarz lemma.

Riemann surfaces

3. In class we have seen that the unit disk $\mathbb{D}$ and the upper half-plane $\mathbb{H}$ are biholomorphically (or conformally) equivalent, and they are both equivalent to the strip $\{z \mid 0 < Im(z) < \pi\}$. Show that the following domains in $\mathbb{C}$ are also conformally equivalent to these three. Note: The Riemann mapping theorem says that every simply connected domain, not all of $\mathbb{C}$, is conformally equivalent to $\mathbb{D}$. The point here is that the conformal equivalence is realized by explicit maps.

   (i) The sector $\{z \mid 0 < arg(z) < \alpha\}$ for any $\alpha \in (0, 2\pi]\}$.

   (ii) Upper half-plane with a slit $\mathbb{H} \setminus [0, i]$. Hint: z^2.

   (iii) Upper half-plane with a circular slit $\mathbb{H} \setminus \{z \mid |z| = 1, Re(z) \geq 0\}$.

   (iv) Half-strip $\{z \mid Re(z) < 0, 0 < Im(z) < \pi\}$. Hint: e^z.

4. Show that $\mathbb{C} \setminus \{0\}$ and $\mathbb{D} \setminus \{0\}$ are not conformally equivalent.

5. Recall from class that

$$X(a,b,c,\infty) = \{(z,w) \in \hat{\mathbb{C}} \times \hat{\mathbb{C}} \mid w^2 = (z-a)(z-b)(z-c)\}$$

is a Riemann surface for $a,b,c \in \hat{\mathbb{C}}$ distinct. It is a double branched cover over $\hat{C}$ branched over a,b,c,∞. Show that if the cross-ratios $(a,b;c,\infty)$ and $(a',b';c',\infty)$ are equal after possibly permuting the points, then $X(a,b,c,\infty)$ and $X(a,b,c,\infty)$ are conformally equivalent. Hint: This is really an exercise in covering spaces. Note: The surface $X(a,b,c,\infty)$ is a torus, and we will see that if the cross-ratios are different then the tori are not biholomorphic. In fact, the "moduli space" of tori (or "elliptic curves" in algebraic geometry) is the space of quadruples of distinct points in the Riemann sphere with the cross-ratio equivalence. Tori, along with annuli, have simplest interesting moduli spaces.

6. The cylinder $\mathbb{C}/2\pi i$ can be identified with $\mathbb{C} \setminus \{0\}$ via $z \mapsto exp(z)$ and then compactified to the Riemann sphere by adding two points at infinity. The cylinder $X = \mathbb{C}/2\pi$ can similarly be compactified by first identifying it with the original cylinder via a rotation. The function $\sin : \mathbb{C} \to \mathbb{C}$ factors through X and gives $S : X \to \mathbb{C}$. Show that S extends to a holomorphic function $\hat{S} : \hat{X} \to \hat{C}$ between the compactifications and then show that $\hat{S}$ is a double branched cover.

A "well-known" identity.

The goal here is to prove that the "principal" sum

$$\sum_{n\in\mathbb{Z}} \frac{1}{z-n} = \pi\cot(\pi z)$$

The sum doesn't converge in the usual sense, just like the harmonic series doesn't, but if we group the terms for n and $-n$ it does:

7. Show that $f(z) = \frac{1}{z} + \sum_{n\geq 1} \frac{2z}{z^2-n^2}$ converges absolutely for every $z \notin \mathbb{Z}$, and f is a meromorphic function on $\mathbb{C}$ with simple poles at $\mathbb{Z}$ and residue 1 at each.

8. Show that $f(z+1) = f(z)$ for every z. Hint: $f(z)$ is the limit of $\sum_{n=-N}^{N} \frac{1}{z-n}$.

9. Let X be the Riemann surface $\mathbb{C}/z \sim z+1$, i.e. a cyclinder. Thus f defines a holomorphic function $F : X \to \hat{\mathbb{C}}$. Let $\hat{X}$ be the compactification of X biholomorphic to $\hat{\mathbb{C}}$, with two ideal points $\pm\infty$

(corresponding to imaginary part going to $\pm\infty$) as in Problem 6. Show that F extends to a holomorphic function $\hat{F} : \hat{X} \to \hat{\mathbb{C}}$ via $\hat{F}(\pm\infty) = \mp\pi i$. Hint: It suffices to extend continuously. When $z = Mi$ this amounts to computing $-2Mi\sum_{n\geq 1}\frac{1}{M^2+n^2}$. The sum can be estimated with $\int_0^\infty \frac{1}{M^2+x^2}dx$, which can be computed via calculus or calculus of residues.

10. Show that $g(z) = \pi\cot(\pi z)$ also has the following properties: it is meromorphic with simple poles at $\mathbb{Z}$ and residue 1 at each, $g(z+1) = g(z)$, and the induced map G on the cylinder X extends to a function $\hat{G}$ on the compactification with the same values at $\pm\infty$.

11. Show that this implies $f = g$. Hint: Show that $\hat{F}\hat{G}^{-1}$ is a Möbius transformation that fixes three points.

There is a more direct way to do this, but I think it involves more calculations. Namely, using the method to solve Basel's problem, we can calculate $f(z)$ for all z. The advantage of the method outlined above is that the only real work is calculating $f(\pm\infty)$, the rest is citing theorems about the Riemann sphere and Möbius transformations.