Homework 6: Review – Problems from past quals

I didn't try to sort these in any way, some are very similar to each other, and there are no hints (with one exception).

1. Let f be a holomorphic function on a disk centered at 0 and radius > 1. Prove that if $|z| < 1$ then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\sin(\xi - z)} d\xi$$

where C is the unit circle oriented counterclockwise.

2. Prove that if f is an entire function satisfying

$$|f(z)| \leq A + B \log |z|$$

for some $A, B > 0$ and all z with $|z| \geq 1$, then f is a constant function.

3. Determine if there exists an entire function f such that

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$$

for all $n = 1, 2, \cdots$.

4. Let

$$f(z) = \frac{e^{\frac{1}{z-1}}}{e^z - 1}$$

Determine all isolated singularities of f and their type. Also compute the residue at each pole.

5. Evaluate the integral

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2a\cos\theta + a^2} d\theta$$

where a is a complex number with $|a| < 1$.

6. Let f be a holomorphic function with an isolated singularity at $z = a$. Assume that $g = \frac{1}{f}$ also has an isolated singularity at $z = a$, and that a is not a removable singularity for either f or g. Determine what type of singularity is a for f and g. Do you know an example of such a function f?

7. Let f be holomorphic in $\mathbb{C} \setminus \{0, 2\}$. Assume:

   (i) 0 and 2 are poles of order 1,

   (ii) f is bounded on $|z| \geq 3$,

   (iii) the integral of f over the circle $C(0,1)$ centered at 0 and radius 1 is $2\pi i$, and

   (iv) the integral of f over the circle $C(0,3)$ centered at 0 and radius 3 is 0.

   Determine f.

8. Let f be an entire function such that $f' = f$. Prove that $f(z) = Ce^z$ for some constant C. Hint: Power series

9. Evaluate the integral
$$\int_0^\infty \frac{\sin ax}{x(x^2 + b^2)} dx$$
   where $a, b \in \mathbb{R}$ and $b \neq 0$.

10. Characterize all entire functions f such that
$$\lim_{z \to \infty} \frac{1}{f(z)} = 0$$

11. Let $\Omega \subseteq \mathbb{C}$ be open and f holomorphic in Ω. Let $b \in \Omega$ and assume $f'(b) \neq 0$. Show that
$$\frac{2\pi i}{f'(b)} = \int_C \frac{1}{f(z) - f(b)} dz$$
   for sufficiently small positively oriented circles C centered at b.

12. Determine the poles and their orders of the function
$$\frac{1}{e^z - 1} - \frac{1}{z}$$

13. Suppose $a > 1$. Compute
$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta}$$

14. Evaluate
$$\int_0^\infty \frac{1}{x^3 + 1} dx$$

15. Suppose f is holomorphic in $0 < r < |z| < R$ and suppose that for some ρ with $r < \rho < R$

$$\int_{|z|=\rho} f(z) z^n dz = 0$$

for all negative integers n. Show that f extends to a holomorphic function on $|z| > r$.

16. Let f be an injective entire function and put $g(z) = f(1/z)$. Show that g does not have an essential singularity at $z = 0$. Further show that $f(z) = az + b$ for some constants a, b.

17. Let f be holomorphic on $D \setminus \{0\}$ where D is the unit disk. Assume that f' is bounded on $D \setminus \{0\}$. Prove that f extends to a holomorphic function on D.

18. Let f be an entire function such that $|f(z)| \neq 1$ for all $z \in \mathbb{C}$. Show that f is a constant function.

19. Let f be an entire function.

    (a) If there is a polynomial g such that $|f(z)| \leq |g(z)|$ for every z, show that f is also a polynomial.

    (b) Can we draw the same conclusion if g is assumed to be a rational function instead of a polynomial?

20. Show that any continuous function on $|z| \leq 1$ which is holomorphic on $|z| < 1$ can be uniformly approximated by polynomials.

21. Suppose f is holomorphic on $|z| < 1$ and satisfies $|f(1/n)| \leq 1/n^n$ for all $n = 2, 3, \cdots$. Determine f.

22. If f is entire and nowhere 0, show that there is an entire function g such that $f = g^2$.

23. If f is entire and not a polynomial, show that for every $\epsilon > 0$ there is z with $|f(z) - z^2| < \epsilon$.